# Optimal wavelet reconstructions from Fourier samples via generalized sampling

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***Abstract*—We consider the problem of computing wavelet coefficients of compactly supported functions from their Fourier samples. For this, we use the recently introduced framework of generalized sampling in the context of compactly supported orthonormal wavelet bases. Our first result demonstrates that using generalized sampling one obtains a stable and accurate reconstruction, provided the number of Fourier samples grows linearly in the number of wavelet coefficients recovered. We also present the exact constant of proportionality for the class of Daubechies wavelets.**

**Our second result concerns the optimality of generalized sampling for this problem. Under some mild assumptions generalized sampling cannot be outperformed in terms of approximation quality by more than a constant factor. Moreover, for the class of so-called perfect methods, any attempt to lower the sampling ratio below a certain critical threshold necessarily results in exponential ill-conditioning. Thus generalized sampling provides a nearly-optimal solution to this problem.**

## I. Generalized sampling

A fundamental problem of signal processing is the reconstruction of signals from a discrete set of measurements. This can be formulated in a Hilbert Space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$, where one seeks to reconstruct a function $f \in \mathcal{H}$ from measurements of the form $\langle f, s_j \rangle$ for some $\{s_j\}_{j\in\mathbb{N}} \subseteq \mathcal{S} \subseteq \mathcal{H}$. A key development is the Shannon-Nyquist Sampling Theorem, which stated that bandlimited or compactly supported signals to be fully described via measurements $\left\langle f, e^{2\pi i \epsilon j \cdot} \right\rangle$, $j \in \mathbb{Z}$, for some appropriate $\epsilon > 0$. In particular, $f$ and its Fourier transform $\hat{f}(\cdot) = \int f(x) e^{-ix\cdot} \mathrm{d}x$ can be approximated respectively as follows:

$$f_N(t) = \epsilon \sum_{|k|\leq N} \hat{f}(2\pi k\epsilon) e^{2\pi i \epsilon k t}, \qquad f_N \xrightarrow{L^2} f,$$

$$\hat{f}_N(t) = \sum_{|k|\leq N} \hat{f}(2\pi k\epsilon) \mathrm{sinc}\left(\frac{t + 2\pi k\epsilon}{2\epsilon}\right), \quad \hat{f}_N \xrightarrow{L^2, L^\infty} \hat{f}.$$

However, in many cases, such approximations are not used because the bases generated by the sinc-function or complex exponentials are generally considered inappropriate representation systems for the underlying signals [1]. In fact, many images and signals can be better represented in terms of a different basis (e.g. splines or wavelets) than the basis in which they are sampled (e.g. the Fourier basis). Consequently, there is much interest in generalising the Shannon-Nyquist Sampling Theorem to recover the coefficients of a signal or image in a particular basis from samples taken with respect to another basis[1], this problem is often referred to as generalized sampling.

The goal now is to reconstruct in an arbitrary space $\mathcal{W} \subseteq \mathcal{H}$ without placing any constraints on the type of input vectors. In practice, we seek an approximation of $f$ in the finite dimensional space $\mathcal{W}_N = \mathrm{span}\{w_j : 1 \leq j \leq N\}$ such that $\overline{\bigcup_{j\in\mathbb{N}} \mathcal{W}_j} = \mathcal{W}$ from some finite set of measurements $\hat{f}_M = (\langle f, s_j \rangle)_{j=1}^M$.

### A. Desirable qualities of the reconstruction algorithm

We will be primarily be concerned with perfect reconstruction algorithms, where the underlying signals can be perfectly reconstructed from our discrete measurement sets. So, if $f \in \mathcal{W}$, then the algorithm should be able to recover $f$ exactly from its measurements. Note that if $\mathcal{W} \cap \mathcal{S}^\perp \neq \{0\}$, then there will exist some non-zero vector $g \in \mathcal{W} \cap \mathcal{S}^\perp$ such that $\langle g, s_j \rangle = 0$ for all $j$. So $g$ is indistinguishable from 0, regardless of the reconstruction algorithm. Thus, when considering the reconstruction problem, we will require that $\mathcal{W}$ and $\mathcal{S}$ satisfy the following condition:

$$\mathcal{W} \cap \mathcal{S}^\perp = \{0\}, \quad \mathcal{W} + \mathcal{S}^\perp \text{ is closed in } \mathcal{H} \tag{1}$$

and will refer to this as the subspace condition. Let us now consider the desirable qualities of a 'good' reconstruction method: Any reconstruction method should be such that the approximation will converge to the true signal as the number of samples increases and the method should be robust to small perturbations in the input data. With this in mind, we consider the following two definitions :

**Definition I.1.** *[2] Let $F_{N,M} : \mathcal{H} \to \mathcal{W}_N$. The quasi-optimality constant $\mu = \mu(F_{N,M})$ is the least constant such that*

$$\|f - F_{N,M}(f)\| \leq \mu \|f - P_{\mathcal{W}_N} f\|, \quad \forall f \in \mathcal{H},$$

*If no such constant exists, we write $\mu = \infty$. We say that $F_{N,M}$ is quasi-optimal if $\mu(F_{N,M})$ is small.*

Note that $P_{\mathcal{W}_N} f$ is the best approximation in norm to $f$ from $\mathcal{W}_N$. So quasi-optimality means that the difference in norm between $f$ and $F_{N,M}(f)$ is at most a constant factor $\mu$ of the difference between $f$ and its best approximation in the subspace $\mathcal{W}_N$.

We also define the condition number of a reconstruction:

**Definition I.2.** *[2] Let $F_{N,M} : \mathcal{H} \to \mathcal{W}_N$ be a mapping such that, for each $f \in \mathcal{H}$, $F_{N,M}(f)$ depends only on the samples $\{\hat{f}_j\}_{j=1}^M$. The condition number of $\kappa(F_{N,M})$ is given by*

$$\kappa(F_{N,M}) = \sup_{f\in\mathcal{H}} \lim_{\epsilon\to 0^+} \sup_{\substack{g\in\mathcal{H} \\ 0<\|\hat{g}\|\leq\epsilon}} \frac{\|F_{N,M}(f+g) - F_{N,M}(f)\|}{\|\hat{g}\|},$$

*where $\hat{g} = \{\hat{g}_j\}_{j=1}^M \in \mathbb{C}^M$. The mapping $F_{N,M}$ is well-conditioned if $\kappa(F_{N,M})$ is small and ill-conditioned otherwise.*

We say that the reconstruction $F_{N,M}$ is 'good' if it is stable and quasi-optimal. In other words, if the *reconstruction constant*

$$C(F_{N,M}) = \max\{\kappa(F_{N,M}), \mu(F_{N,M})\},$$

is small.

## II. Reduced consistency sampling

This problem of generalized sampling is not new and has been extensively studied - important contributions include the consistent sampling scheme introduced by Aldroubi and Unser [3], [4], [5] and significantly extended by Eldar [6], [7].

For $f \in \mathcal{H}$, one seeks an approximation $F_N(f) \in \mathcal{W}_N$ which agrees with the given measurements, so it is such that

$$\langle F_N(f), s_j \rangle = \langle f, s_j \rangle, \quad j = 1, \ldots, N. \tag{2}$$

This involves solving a linear system of $N$ equations and $F_N(f)$ exists uniquely when $\mathcal{W}_N \oplus \mathcal{S}_N^{\perp} = \mathcal{H}$. However, this condition need not hold even if $\mathcal{W} \oplus \mathcal{S}^{\perp} = \mathcal{H}$ and there are important cases for which (2) has no solution, or the method $F_N$ is unstable or nonconvergent, i.e. $\kappa(F_N) \to \infty$ or $F_N(f) \not\to f$ as $N \to \infty$ [8], [5].

To circumvent these problems, various authors have considered overdetermined systems, where the number of measurements exceeds the number of reconstruction coefficients to be recovered. We in particular mention the work of Pruessmann et al [9] in the recovery of voxel coefficients (which may be considered as Haar wavelet coefficients) from Fourier samples and [10] by Hrycak and Gröchenig for the recovery of polynomial coefficients from Fourier samples. To formalise these approaches, Adcock and Hansen introduced the reduced consistency sampling scheme [8], [11]. The task is then as follows: Given $N \in \mathbb{N}$, for some appropriate $M \in \mathbb{N}$, find $F_{N,M}(f) \in \mathcal{W}_N$ such that

$$\langle P_{\mathcal{S}_M} F_{N,M}(f), w_j \rangle = \langle P_{\mathcal{S}_M} f, w_j \rangle, j = 1, \ldots, N. \tag{3}$$

So, $F_{N,M}(f)$ coincides with $f$ on $P_{\mathcal{S}_M}(\mathcal{W}_N)$ rather than on $\mathcal{S}_M$. Under this framework, a stable and convergent scheme can always be devised. Indeed, for all $N \in \mathbb{N}$, there exists $m_0$ such that for all $M \geq m_0$, there exists a unique $F_{N,M}(f)$ satisfying (3), and such a solution is quasi-optimal in $\mathcal{W}_N$ and stable with reconstruction constant at most

$$D_{N,M} = \left( \inf_{g \in \mathcal{W}_N} \| P_{\mathcal{S}_M} g \| \right)^{-1}.$$

As both convergence and numerical stability are governed by the quantity $D_{N,M}$, the notion of a *stable sampling rate* was introduced:

**Definition II.1.** *[2] For $N \in \mathbb{N}$ and $\theta \in (1, \infty)$, the stable sampling rate is given by*

$$\Theta(N; \theta) = \min \{ M \in \mathbb{N} : D_{N,M} \leq \theta \}.$$

As demonstrated in [2], for any $N \in \mathbb{N}$, $\Theta(N;\theta)$ can be numerically calculated and determines the number of samples required to obtain a convergent and stable reconstructions in $\mathcal{W}_N$ as $N \to \infty$.

## III. Optimality of generalized sampling

In [2], the reduced consistency scheme is shown to be optimal amongst all perfect methods, in that it is not possible improve upon its stability. The following result shows that the stable sampling rate is a universal property amongst perfect methods, since any perfect method must sample at a rate at least that of the stable sampling rate to achieve the same stability.

**Theorem III.1.** *[2] For $M \geq N$ let $G_{N,M} : \mathcal{H} \to \mathcal{W}_N$ be a perfect reconstruction method such that, for each $f \in \mathcal{H}$, $G_{N,M}(f)$ depends only on the samples $\{\hat{f}_j\}_{j=1}^M$. Then the condition number is such that $\kappa(G_{N,M}) \geq \kappa(F_{N,M})$, where $F_{N,M}$ is the generalized sampling reconstruction.*

For nonperfect methods, the following result holds:

**Theorem III.2.** *[2] Suppose that the stable sampling rate $\Theta(N;\theta)$ is linear in $N$ for a particular sampling and reconstruction problem. Let $f \in \mathcal{H}$ be fixed, and suppose that there exists a sequence of mappings*

$$G_M : \{\hat{f}_j\}_{j=1}^M \mapsto G_M(f) \in \mathcal{W}_{\Psi_f(M)},$$

*where $\Psi_f : \mathbb{N} \to \mathbb{N}$ with $\Psi_f(M) \leq cM$. Suppose also that there exist constants $c_1(f), c_2(f), \alpha_f > 0$ such that*

$$c_1(f) N^{-\alpha_f} \leq \| f - P_{\mathcal{W}_N} f \| \leq c_2(f) N^{-\alpha_f}, \quad \forall N \in \mathbb{N}. \tag{4}$$

*Then, given $\theta \in (1, \infty)$, there exist constants $c(\theta) \in (0,1)$ and $c_f(\theta) > 0$ such that*

$$\| f - F_{c(\theta)M,M}(f) \| \leq c_f(\theta) \| f - G_M(f) \|, \quad \forall M \in \mathbb{N}, \tag{5}$$

*where $F_{N,M}$ is the generalized sampling reconstruction.*

Thus, for problems with linear stable sampling rates, even if one is allowed to design a method that depends on $f$ in a completely non-trivial way, it is still not possible to obtain a faster asymptotic rate of convergence than that of generalized sampling. In fact, we will show that the stable sampling rate is linear for wavelets, making this theorem directly applicable.

## IV. Wavelet reconstructions from Fourier samples

Any implementation of the reduced consistency sampling scheme requires an understanding of the corresponding stable sampling rate. The case where the reconstruction space $\mathcal{W}$ is generated by compactly supported wavelets and the sampling space is the space of complex exponentials $\mathcal{S} = \overline{\text{span}} \left\{ e^{2\pi i \epsilon j \cdot} : j \in \mathbb{Z} \right\}$ for some appropriate $\epsilon > 0$ is particularly important, with applications in medical imaging. In this section, we present some results which show that the stable sampling rate is linear in this setting. We first describe the construction of the reconstruction and sampling spaces.

For the reconstruction space, we aim to create orthonormal subsets $\{\varphi_k\}_{k \in \mathbb{N}} \subseteq L^2(\mathbb{R})$ with the property that $L^2[0,a] \subseteq \overline{\text{span}} \{\varphi_k\}_{k \in \mathbb{N}}$ for some $a > 0$. Suppose that we are given an orthonormal mother wavelet $\psi$ and an orthonormal scaling function $\phi$ such that $\text{supp}(\psi) = \text{supp}(\phi) = [0, a]$ for some $a \geq 1$.

The standard approach is to consider the following collection of functions

$$\Omega_a = \{ \phi_k, \psi_{j,k} : \text{supp}(\phi_k)^o \cap [0,a] \neq \emptyset, \\ \text{supp}(\psi_{j,k})^o \cap [0,a] \neq \emptyset, j \in \mathbb{Z}_+, k \in \mathbb{Z}, \},$$

where

$$\phi_k = \phi(\cdot - k), \qquad \psi_{j,k} = 2^{\frac{j}{2}} \psi(2^j \cdot - k).$$

(the notation $K^o$ denotes the interior of a set $K \subseteq \mathbb{R}$). This now gives

$$L^2[0,a] \subseteq \text{cl}(\text{span}\{\varphi : \varphi \in \Omega_a\}) = \mathcal{W} \subseteq L^2[-T_1, T_2],$$

where $T_1 = \lceil a \rceil - 1$ and $T_2 = 2\lceil a \rceil - 1$ are such that $[-T_1, T_2]$ contains the support of all functions in $\Omega_a$.

For the Fourier sampling space, we let $\epsilon \leq 1/(T_1 + T_2)$ be the *sampling density*. Note that $1/(T_1 + T_2)$ is the corresponding Nyquist criterion for functions supported on $[-T_1, T_2]$. We now define the sampling vectors by

$$s_l = \sqrt{\epsilon} e^{2\pi i l \epsilon \cdot} \chi_{[-T_1/(\epsilon(T_1+T_2)), T_2/(\epsilon(T_1+T_2))]},$$

and the sampling space by

$$\mathcal{S} = \overline{\operatorname{span}\{s_l : l \in \mathbb{Z}\}}$$
$$= \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp}(f) \subseteq \left[ -\frac{T_1}{\epsilon(T_1+T_2)}, \frac{T_2}{\epsilon(T_1+T_2)} \right] \right\}$$

and the space spanned by the first $M$ sampling vectors by

$$\mathcal{S}_M = \operatorname{span}\left\{ s_l : -\left\lfloor \frac{M}{2} \right\rfloor \leq l \leq \left\lceil \frac{M}{2} \right\rceil - 1 \right\}.$$

Our main result on the stable sampling rate is as follows.

**Theorem IV.1.** *[12] For $R \in \mathbb{N}$, let $N_R$ denote the number of elements in $\Omega_a$ of the form $\phi_{j,k}$ or $\psi_{j,k}$ with $j < R$, in particular, $N_R = 2^R\lceil a \rceil + (R+1)(\lceil a \rceil - 1)$. Then for $N \leq N_R$ and all $\theta \in (1,\infty)$, there exists $S_\theta \in \mathbb{N}$, independent of $R$, such that for $M = \left\lceil \frac{S_\theta 2^{R+1}}{\epsilon} \right\rceil$, we have $D_{NM} \leq \theta$. Hence, $\Theta(N,\theta) = \mathcal{O}(N)$ for any $\theta \in (1,\infty)$.*

So, the stable sampling rate is linear and in other words, given any $f \in \mathcal{H}$, for any $N \in \mathbb{N}$ and $\theta \in (1,\infty)$, there exists a constant $r$ such that $r \cdot N$ samples will up to a factor of $\theta$, yield the best possible approximation in the space $\mathcal{W}_N$ and the condition number of the method is no worse than $\theta$ as $N \to \infty$.

One may ask, how small can the ratio $r$ be? The next result shows that there is a critical ratio, below which, the reconstruction will become exponentially ill posed.

**Theorem IV.2.** *[12] Let $F_{N,M}$ denote the reduced consistency sampling method and $N_R$ be as in Theorem IV.1. Let $N = N_R$ and $M = c \cdot 2^R$, with $c < \epsilon^{-1}$. Then $\kappa(F_{N,M}) \to \infty$ exponentially as $N \to \infty$.*

The first consequence of this with regards to optimality is that this critical ratio is universal amongst perfect methods. It is not the case that a perfect method could reconstruct in $\mathcal{W}_{N_R}$ from less than $2^R/\epsilon$ samples and still only experience mild growth in its condition number - this method will inherently become exponentially ill posed. The second consequence for optimality is as explained at the end of Section III, any non-perfect method which has a lower sampling ratio for a particular function $f$ satisfying (4) can only outperform generalized sampling by a constant factor.

### A. Daubechies wavelets

Our next result examines the special case of Daubechies wavelets and asymptotically, the stable sampling ratio can be determined exactly.

**Theorem IV.3.** *[12] Let $\mathcal{W}$ be generated by a Daubechies wavelet, and recall $N_R$ from Theorem IV.1. Then, there exists $\theta \in (1,\infty)$ and $R_0 \in \mathbb{N}$ such that for all $R \geq R_0$, $\Theta(N_R,\theta) = \lceil 2^R/\epsilon \rceil$. In particular, when $1/\epsilon \in \mathbb{Z}$ it suffices to let $\theta > \left( \inf_{\xi \in [-\pi,\pi]} \left| \hat{\phi}(\xi) \right| \right)^{-1}$. Moreover, in addition to this, for Haar wavelets, where $a = 1$, we have that $\Theta(N_R,\theta) \leq \lceil 2^R/\epsilon \rceil$ for all $R \in \mathbb{N}$.*

## V. Numerical examples

In this section, we provide numerical simulations of three key ideas for generalized sampling in the context of wavelet reconstructions from Fourier samples. Firstly, generalized sampling can offer substantial improvements. Secondly, the stable sampling rate is linear for wavelet reconstructions from Fourier samples, moreover, our result for the Daubechies wavelet case is sharp. Finally, understanding of the stable sampling rate is crucial to the implementation of reduced consistency sampling and violation of it could lead to disastrous results.

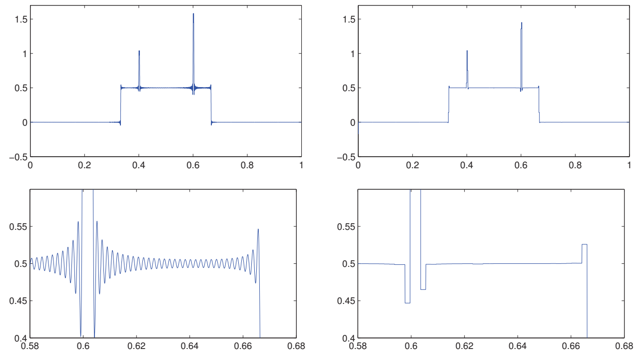

Fig. 1. The top row shows $f_M$ (left) and $f^{[N,M]}$ (right). The bottom row shows $f_M$ (left) and $f^{[N,M]}$ (right) on the interval $[0.58, 0.68]$.

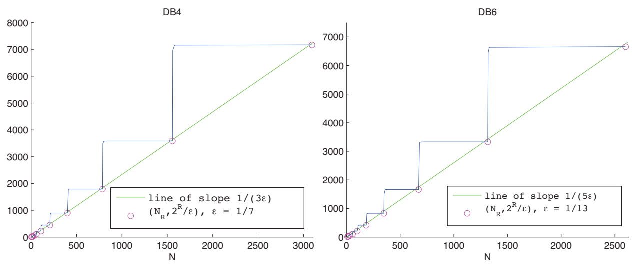


Fig. 2. The figure displays the stable sampling rate $\Theta(N,\theta_1)$ and $\Theta(N,\theta_2)$ in blue for the Daubechies-4 wavelet (left) and the Daubechies-6 wavelet (right) with Fourier samples at a sampling distance $\epsilon = 1/7$ and $\epsilon = 1/13$ respectively.

### A. Signal recovery via generalized sampling

We consider the reconstruction of the following function

$$f = \frac{1}{2}\chi_{[1/3,2/3]} + \frac{1}{2}\chi_{[2/5,2/5+1/300]} + \chi_{[3/5,3/5+1/300]},$$

from $M = 1024$ Fourier samples of sampling density $\epsilon = 1/2$. Figure 1 shows the truncated Fourier series representation $f_M$ as presented in the S-N Sampling Theorem as well as the reconstruction $f^{[N,M]}$ from implementing generalized sampling for a Haar wavelet reconstruction space. In this case, $N$ is chosen to be 512. It is clear that $f^{[N,M]}$ is visually preferable to $f_M$ with less oscillations at discontinuities. We remark that similar figures were generated in [13] to justify the use of wavelet encoding for MRI, which modifies an MR scanner to direct acquire wavelet coefficients rather than Fourier samples. In proving that the stable sampling rate is linear, we show that that wavelet coefficients can be accurately approximated via a post-processing and there is little to be gained in modifying the sampling process.

### B. Sharpness of Theorem IV.3

To demonstrate the sharpness of this result, we consider the Daubechies-4 wavelet (supported in $[0,3]$), and the Daubechies-6 wavelet (supported in $[0,5]$). The graphs of Figure 2 plots the stable sampling rate $\Theta(N,\theta)$ against $N$, the number of reconstruction vectors to be recovered. In each case, we set $\theta > \left( \inf_{\xi \in [-\pi,\pi]} \left| \hat{\phi}(\xi) \right| \right)^{-1}$. Note that at the points $N_R$, $\Theta(N_R,\theta) = 2^R/\epsilon$ as predicted by Theorem IV.3.

| $M$ | $\|f-\tilde{f}_{M/c,M}\|_{L^2}$ | $\|f-\tilde{f}_{M/c_1,M}\|_{L^2}$ | Noise Level |
|---|---|---|---|
| 482 | $7.3\times10^{-7}$ | $2.8\times10^{-2}$ | 0 |
| 934 | $1.4\times10^{-7}$ | $5.4\times10^{-2}$ | 0 |
| 1834 | $2.6\times10^{-8}$ | $1.4\times10^{-2}$ | 0 |
| 482) | $9.6\times10^{-6}$ | 6.1 | $1.0\times10^{-5}$ |
| 934 | $9.5\times10^{-6}$ | 77.2 | $1.0\times10^{-5}$ |
| 1834 | $9.7\times10^{-6}$ | 85.9 | $1.0\times10^{-5}$ |

TABLE I

THE TABLE SHOWS THE ERROR OF THE GENERALIZED SAMPLING RECONSTRUCTIONS $\tilde{f}_{N,M}$ WITH $N = M/c$ AND $N = M/c_1$, WITH NOISELESS AND NOISY DATA.

Observe also from Theorem IV.3 that

$$\Theta(N_R,\theta) < \Theta(N,\theta) \leq \Theta(N_{R+1},\theta), \quad N_R < N \leq N_{R+1}.$$

The staircase effect witnessed in the figure suggests that the upper bound is in fact an equality. Hence, although the stable sampling rate is linear for all $N$, from the point of view of the stable sampling rate at least, there is nothing to be gained from allowing $N \neq N_R$.

### C. Importance of the stable sampling rate

We demonstrate, as predicted by Theorem IV.2, that failure of satisfying the stable sampling rate gives a completely unstable and non-convergent reconstruction. We compare the choices

$$M = cN, \quad c = \frac{1}{\epsilon\lceil a\rceil}, \qquad M = c_1 N, \quad c_1 = 0.95c.$$

for the recovery of the function $f = \sum_{j=1}^{3\times10^3} j^{-3}\varphi_j$, where $\varphi_j$ are Daubechies$-4$ wavelets. We will consider Fourier samples $\langle f, s_j\rangle$ for $|j| \leq M/2$ which are contaminated with noise and thus we observe $\xi = \{\langle f, s_1\rangle, \dots, \langle f, s_M\rangle\} + v$ with $\|v\| = \varepsilon$ for some noise level $\varepsilon \geq 0$. As verified in Table I the latter choice of $M = c_1N$ gives disastrous results as an incorrect choice of the sampling ratio causes the condition number of the algorithm to blow up exponentially.

## VI. EXTENSION TO OTHER MRA WAVELET BASES

Although the theorems presented in the previous sections have been for orthonormal systems of MRA wavelets, the key property required for the proofs is the existence of an increasing sequence

$$0 < N_1 < \cdots < N_R < N_{R+1} < \cdots$$

such that $N_R = \mathcal{O}(2^R)$, $\overline{\bigcup_{R\in\mathbb{N}} \mathcal{W}_{N_R}} = \mathcal{W}$ and

$$\begin{aligned} &\mathcal{W}_{N_R} \subseteq \operatorname{span}\{\phi_{R,j} : A_{R,1} \leq j \leq A_{R,2}\}, \\ &A_{R,2} - A_{R,1} = \mathcal{O}(2^R). \end{aligned} \tag{6}$$

Consequently, the results of this paper can be readily extended to other compactly supported MRA wavelets such as the Semi-orthogonal spline wavelets of [14], [15] or the bi-orthogonal Cohen-Daubechies-Feauveau wavelets of [16]. We also remark that the construction of the wavelet reconstruction space in Section IV is the standard construction of wavelets on an interval with zero-padding which can lead to large wavelet coefficients at the end points of the interval. However, there are more sophisticated constructions of wavelets on the intervals to reduce this effect, such as the basis of Daubechies wavelets with special boundary wavelet and scaling functions as described in [17]. Their construction is such that the number of vanishing moments is preserved and the boundary scaling function can be written as a linear combination of finitely many elements in $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$. Such a wavelet basis will also satisfy the requirements of (6) and the associated stable sampling rate is also linear. In combination with known results [18] about the characterization of the Sobolev space $W^s[0,1]$, $s > 0$ via the decay of wavelet coefficients from interval wavelets with $q > s$ vanishing moments, we have the following result.

**Theorem VI.1.** *Let $\mathcal{W}$ be the reconstructed space constructed from the Daubechies wavelet of $q$ vanishing moments on the unit interval and let $\mathcal{S}$ be the Fourier sampling space with sampling density $\epsilon \leq 1$. Then, for any $\theta \in (1,\infty)$, the stable sampling rate $\Theta(N,\theta)$ is linear in $N$. Furthermore, given any $f \in W^s[0,1]$ with $s \in (0,q)$, the generalized sampling solution $F^{[N,M]}(f)$ implemented with $M = \Theta(N,\theta)$ samples satisfies*

$$\left\| f - F^{[N,M]}(f) \right\| = \mathcal{O}(M^{-s}).$$

Thus, another consequence of a linear stable sampling rate is as follows: given $M$ Fourier samples of any $f \in W^s[0,1]$, it is well known that the Fourier representation cannot yield a convergence rate of $\mathcal{O}(M^{-s})$. However, this convergence rate can be attained from exactly these $M$ Fourier measurements by reconstructing in an appropriate wavelet basis via generalized sampling.

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