Breaking the coherence barrier: A new theory for compressed sensing

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1 Introduction

This paper provides an important extension of compressed sensing which bridges a substantial gap between existing theory and its current use in real-world applications.

Compressed sensing (CS), introduced by Candès, Romberg & Tao [16] and Donoho [28], has been one of the major developments in applied mathematics in the last decade [12, 31, 30, 24, 34, 35, 36]. Subject to appropriate conditions, it allows one to circumvent the traditional barriers of sampling theory (e.g. the Nyquist rate), and thereby recover signals from far fewer measurements than is classically considered possible. This has important implications in many practical applications, and for this reason compressed sensing has, and continues to be, very intensively researched.

The theory of compressed sensing is based on three fundamental concepts: *sparsity*, *incoherence* and *uniform random subsampling*. Whilst there are examples where these apply, in many applications one or more of these principles may be lacking. This includes virtually all of medical imaging – Magnetic Resonance Imaging (MRI), Computerized Tomography (CT) and other versions of tomography such as Thermoacoustic, Photoacoustic or Electrical Impedance Tomography – most of electron microscopy, as well as seismic tomography, fluorescence microscopy, Hadamard spectroscopy and radio interferometry. In many of these problems, it is the principle of incoherence that is lacking, rendering the standard theory inapplicable. Despite this issue, compressed sensing has been, and continues to be, used with great success in many of these areas. Yet, to do so it is typically implemented with sampling patterns that differ substantially from the uniform subsampling strategies suggested by the theory. In fact, in many cases uniform random subsampling yields highly suboptimal numerical results.

The standard mathematical theory of compressed sensing has now reached a mature state. However, as this discussion attests, there is a substantial, and arguably widening gap between the theoretical and applied sides of the field. New developments and sampling strategies are increasingly based on empirical evidence lacking mathematical justification. Furthermore, in the above applications one also witnesses a number of intriguing phenomena that are not explained by the standard theory. For example, in such problems, the optimal sampling strategy depends not just on the overall sparsity of the signal, but also on its structure, as will be documented thoroughly in this paper. This phenomenon is in direct contradiction with the usual sparsity-based theory of compressed sensing. Theorems that explain this observation – i.e. that reflect how the optimal subsampling strategy depends on the structure of the signal – do not currently exist.

The purpose of this paper is to provide a bridge across this divide. It does so by generalizing the three traditional pillars of compressed sensing to three new concepts: *asymptotic sparsity*, *asymptotic incoherence* and *multilevel random subsampling*. This new theory shows that compressed sensing is also possible, and reveals several advantages, under these substantially more general conditions. Critically, it also addresses the important issue raised above: the dependence of the subsampling strategy on the structure of the signal.

The importance of this generalization is threefold. First, as will be explained, real-world inverse problems are typically not incoherent and sparse, but asymptotically incoherent and asymptotically sparse. This paper provides the first comprehensive mathematical explanation for a range of empirical usages of compressed sensing in applications such as those listed above. Second, in showing that incoherence is not a requirement for compressed sensing, but instead that asymptotic incoherence suffices, the new theory offers markedly greater flexibility in the design of sensing mechanisms. In the future, sensors need only satisfy this significantly more relaxed condition. Third, by using asymptotic incoherence and multilevel sampling to exploit not just sparsity, but also structure, i.e. asymptotic sparsity, the new theory paves the way for improved recovery algorithms that achieve better reconstructions in practice from fewer measurements.

A critical aspect of many real-world problems such as those listed above is that they do not offer the freedom to design or choose the sensing operator, but instead impose it (e.g. Fourier sampling in MRI). As such, much of the existing compressed sensing work, which relies on random or custom designed sensing matrices, typically to provide universality, is not applicable. This paper shows that in many such applications the imposed sensing operators are highly non-universal and coherent with popular sparsifying bases. Yet they are asymptotically incoherent, and thus fall within the remit of the new theory. Spurred by this observation, this paper also raises the question of whether universality is actually desirable in practice, even in applications where there is flexibility to design sensing operators with this property (e.g. in compressive imaging). The new theory shows that asymptotically incoherent sensing and multilevel sampling allow one to exploit structure, not just sparsity. Doing so leads to notable advantages over universal operators, even for problems where the latter are applicable. Moreover, and crucially, this can be done in a computationally efficient manner using fast Fourier or Hadamard transforms.

This aside, another outcome of this work is that the Restricted Isometry Property (RIP), although a popular tool in compressed sensing theory, is of little relevance in many practical inverse problems. As confirmed later via the so-called *flip test*, the RIP does not hold in many real-world problems.

Before we commence with the remainder of this paper, let us make one further remark. Many of the problems listed above are analog, i.e. they are modelled with continuous transforms, such as the Fourier or Radon transforms. Conversely, the standard theory of compressed sensing is based on a finite-dimensional model. Such *mismatch* can lead to critical errors when applied to real data arising from continuous models, or inverse crimes when the data is inappropriately simulated [18, 41]. To overcome this issue, a theory of compressed sensing in infinite dimensions was recently introduced in [1]. This paper fundamentally extends [1] by presenting the new theory in both the finite- and infinite-dimensional settings, the infinite-dimensional analysis also being instrumental for obtaining the Fourier and wavelets estimates in §6.

2 The need for a new theory

We now ask the following question: does the standard theory of compressed sensing explain its empirical success in the aforementioned applications? We now argue that the answer is no. Specifically, even in fundamental applications such as MRI (recall that MRI was one of the first applications of compressed sensing, due to the pioneering work of Lustig et al. [51, 53, 54, 55]), there is a significant gap between theory and practice.

2.1 Compressed sensing

Let us commence with a short review of finite-dimensional compressed sensing theory – infinite-dimensional compressed sensing will be considered in §5. A typical setup, and one which we shall follow in part of this paper, is as follows. Let $\{\psi_j\}_{j=1}^N$ and $\{\varphi_j\}_{j=1}^N$ be two orthonormal bases of $\mathbb{C}^N$, the *sampling* and *sparsity* bases respectively, and write $U = (u_{ij})_{i,j=1}^N \in \mathbb{C}^{N\times N}$, $u_{ij} = \langle \varphi_j, \psi_i \rangle$. Note that U is an isometry, i.e. $U^*U = I$.

Definition 2.1. *Let $U = (u_{ij})_{i,j=1}^N \in \mathbb{C}^{N\times N}$ be an isometry. The coherence of U is precisely*

$$\mu(U) = \max_{i,j=1,\ldots,N} |u_{ij}|^2 \in [N^{-1}, 1]. \tag{2.1}$$

We say that U is perfectly incoherent if $\mu(U) = N^{-1}$.

A signal $f \in \mathbb{C}^N$ is said to be s-sparse in the orthonormal basis $\{\varphi_j\}_{j=1}^N$ if at most s of its coefficients in this basis are nonzero. In other words, $f = \sum_{j=1}^N x_j \varphi_j$, and the vector $x \in \mathbb{C}^N$ satisfies $|\mathrm{supp}(x)| \leq s$, where $\mathrm{supp}(x) = \{j : x_j \neq 0\}$. Let $f \in \mathbb{C}^N$ be s-sparse in $\{\varphi_j\}_{j=1}^N$, and suppose we have access to the samples $\hat{f}_j = \langle f, \psi_j \rangle$, $j = 1, \ldots, N$. Let $\Omega \subseteq \{1, \ldots, N\}$ be of cardinality m and chosen uniformly at random. According to a result of Candès & Plan [14] and Adcock & Hansen [1], f can be recovered exactly with probability exceeding $1 - \epsilon$ from the subset of measurements $\{\hat{f}_j : j \in \Omega\}$, provided

$$m \gtrsim \mu(U) \cdot N \cdot s \cdot \big(1 + \log(\epsilon^{-1})\big) \cdot \log N, \tag{2.2}$$

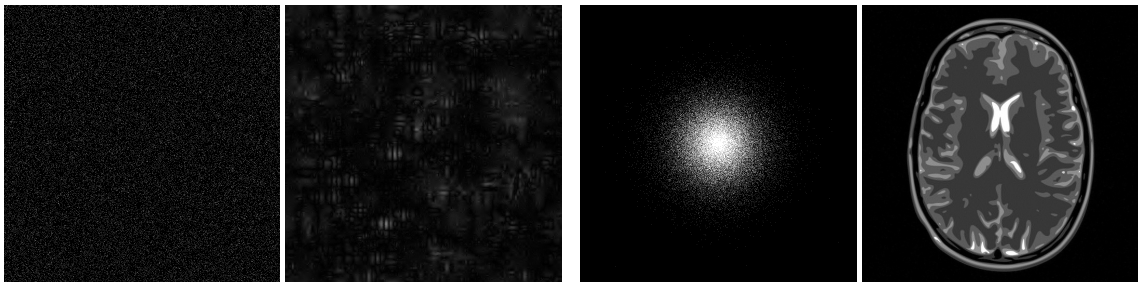


Figure 1: Left to right: (i) 5% uniform random subsampling scheme, (ii) CS reconstruction from uniform subsampling, (iii) 5% multilevel subsampling scheme, (iv) CS reconstruction from multilevel subsampling.

(here and elsewhere in this paper we shall use the notation $a \gtrsim b$ to mean that there exists a constant $C > 0$ independent of all relevant parameters such that $a \geq Cb$). In practice, recovery is achieved by solving the following convex optimization problem:

$$\min_{\eta \in \mathbb{C}^N} \|\eta\|_{l^1} \text{ subject to } P_\Omega U \eta = P_\Omega \hat{f}, \tag{2.3}$$

where $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_N)^\top$ and $P_\Omega \in \mathbb{C}^{N \times N}$ is the diagonal projection matrix with j^{th} entry 1 if $j \in \Omega$ and zero otherwise. The key estimate (2.2) shows that the number of measurements m required is, up to a log factor, on the order of the sparsity s, provided the coherence $\mu(U) = \mathcal{O}\left(N^{-1}\right)$. This is the case, for example, when U is the DFT matrix; a problem which was studied in some of the first papers on compressed sensing [16].

2.2 Incoherence is rare in practice

To test the practicality of the incoherence condition, let us consider a typical compressed sensing problem. In a number of important applications, not least MRI, the sampling is carried out in the Fourier domain. Since images are sparse in wavelets, the usual CS setup is to form the a matrix $U = U_N = U_{\text{df}} V_{\text{dw}}^{-1} \in \mathbb{C}^{N \times N}$, where U_{df} and V_{dw} represent the discrete Fourier and wavelet transforms respectively. However, in this case the coherence

$$\mu(U_N) = \mathcal{O}(1), \quad N \to \infty,$$

for any wavelet basis. Thus, up to a constant factor, this problem has the worst possible coherence. The standard compressed sensing estimate (2.2) states that $m = N$ samples are needed in this case (i.e. full sampling), even though the object to recover is typically highly sparse. Note that this is not an insufficiency of the theory. If uniform random subsampling is employed, then the lack of incoherence leads to a very poor reconstruction. This can be seen in Figure 1.

The underlying reason for this lack of incoherence can be traced to the fact that this finite-dimensional problem is a discretization of an infinite-dimensional problem. Specifically,

$$\underset{N \to \infty}{\text{WOT-lim}}\ U_{\text{df}} V_{\text{dw}}^{-1} = U, \tag{2.4}$$

where $U : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is the operator represented as the infinite matrix

$$U = \begin{pmatrix} \langle \varphi_1, \psi_1 \rangle & \langle \varphi_2, \psi_1 \rangle & \cdots \\ \langle \varphi_1, \psi_2 \rangle & \langle \varphi_2, \psi_2 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.5}$$

and the functions φ_j are the wavelets used, the ψ_j's are the complex exponentials and WOT denotes the weak operator topology. Since the coherence of the infinite matrix U – i.e. the supremum of its entries in absolute value – is a fixed number, we cannot expect incoherence of the discretization U_N for large N: at some point, one will always encounter the *coherence barrier*.

This problem is not isolated to this example. Heuristically, any problem that arises as a discretization of an infinite-dimensional problem will suffer from the same phenomenon. The list of applications of this

type is long, and includes for example, MRI, CT, microscopy and seismology. To mitigate this problem, one may naturally try to change $\{\varphi_j\}$ or $\{\psi_j\}$. However, this will delivery only marginal benefit, since (2.4) demonstrates that the coherence barrier will always occur for large enough N. One may now wonder how it is possible that compressed sensing is applied so successfully to many such problems. The key to this is so-called *asymptotic* incoherence (see §3.1) and the use of a variable density/multilevel subsampling strategy. The success of such subsampling is confirmed numerically in Figure 1. However, it is important to note that this is an empirical solution to the problem. None of the usual theory explains the success of compressed sensing when implemented in this way.

2.3 Sparsity and the flip test

The previous discussion demonstrates that we must dispense with the principles of incoherence and uniform random subsampling in order to develop a new theory of compressed sensing. We now claim that sparsity must also be replaced with a more general concept. This may come as a surprise to the reader, since sparsity is a central pillar of not just compressed sensing, but much of modern signal processing. However, as we now describe, this can be confirmed by a simple experiment we refer to as the *flip test*.

Sparsity asserts that an unknown vector x has s important coefficients, where the locations can be arbitrary. CS establishes that all s-sparse vectors can be recovered in the incoherent setting from the same sampling strategy, i.e. uniform random subsampling. In particular, the sampling strategy is completely independent of the location of these coefficients. The flip test, described next, allows one to evaluate whether this holds in practice. Let $x \in \mathbb{C}^N$ and a measurement matrix $U \in \mathbb{C}^{N \times N}$. Next we take samples according to some appropriate subset $\Omega \subseteq \{1, \ldots, N\}$ with $|\Omega| = m$, and solve:

$$\min \|z\|_1 \text{ subject to } P_\Omega U z = P_\Omega U x. \tag{2.6}$$

This gives a reconstruction $z = z_1$. Now we flip x through the operation

$$x \mapsto x^{\mathrm{fp}} \in \mathbb{C}^N, \quad x_1^{\mathrm{fp}} = x_N,\, x_2^{\mathrm{fp}} = x_{N-1}, \ldots, x_N^{\mathrm{fp}} = x_1,$$

giving a new vector x^{fp} with reverse entries. We now apply the same compressed sensing reconstruction to x^{fp}, using the same matrix U and the same subset Ω. That is we solve

$$\min \|z\|_1 \text{ subject to } P_\Omega U z = P_\Omega U x^{\mathrm{fp}}. \tag{2.7}$$

Let z be a solution of (2.7). In order to get a reconstruction of the original vector x, we perform the flipping operation once more and form the final reconstruction $z_2 = z^{\mathrm{fp}}$.

Suppose now that Ω is a good sampling pattern for recovering x using the solution z_1 of (2.6). If sparsity is the key structure that determines such reconstruction quality, then we expect exactly the same quality in the approximation z_2 obtained via (2.7), since x^{fp} is merely a permutation of x. Is this true in practice? To illustrate this we consider several examples arising from the following applications: fluorescence microscopy, compressive imaging, MRI, CT, electron microscopy and radio interferometry. These examples are based on the matrix $U = U_{\mathrm{dft}} V_{\mathrm{dwt}}^{-1}$ or $U = U_{\mathrm{Had}} V_{\mathrm{dwt}}^{-1}$, where U_{dft} is the discrete Fourier transform, U_{Had} is a Hadamard matrix and V_{dwt} is the discrete wavelet transform.

The results of this experiment are shown in Figure 2. As is evident, in all cases the flipped reconstructions z_2 are substantially worse than their unflipped counterparts z_1. Hence, sparsity alone does not govern the reconstruction quality, and consequently the success in the unflipped case must also be due in part to the structure of the signal. We therefore conclude the following:

The success of compressed sensing depends critically on the structure of the signal.

Put another way, the optimal subsampling strategy Ω depends on the signal structure.

Note that the flip test reveals another interesting phenomenon:

There is no Restricted Isometry Property (RIP).

Suppose the matrix $P_\Omega U$ satisfied an RIP for realistic parameter values (i.e. problem size N, subsampling percentage m, and sparsity s) found in applications. Then this would imply recovery of all approximately sparse vectors with the same error. This is in direct contradiction with the results of the flip test.

Note that in all the examples in Figure 2, uniform random subsampling would have given nonsensical results, analogously to what was shown in Figure 1.

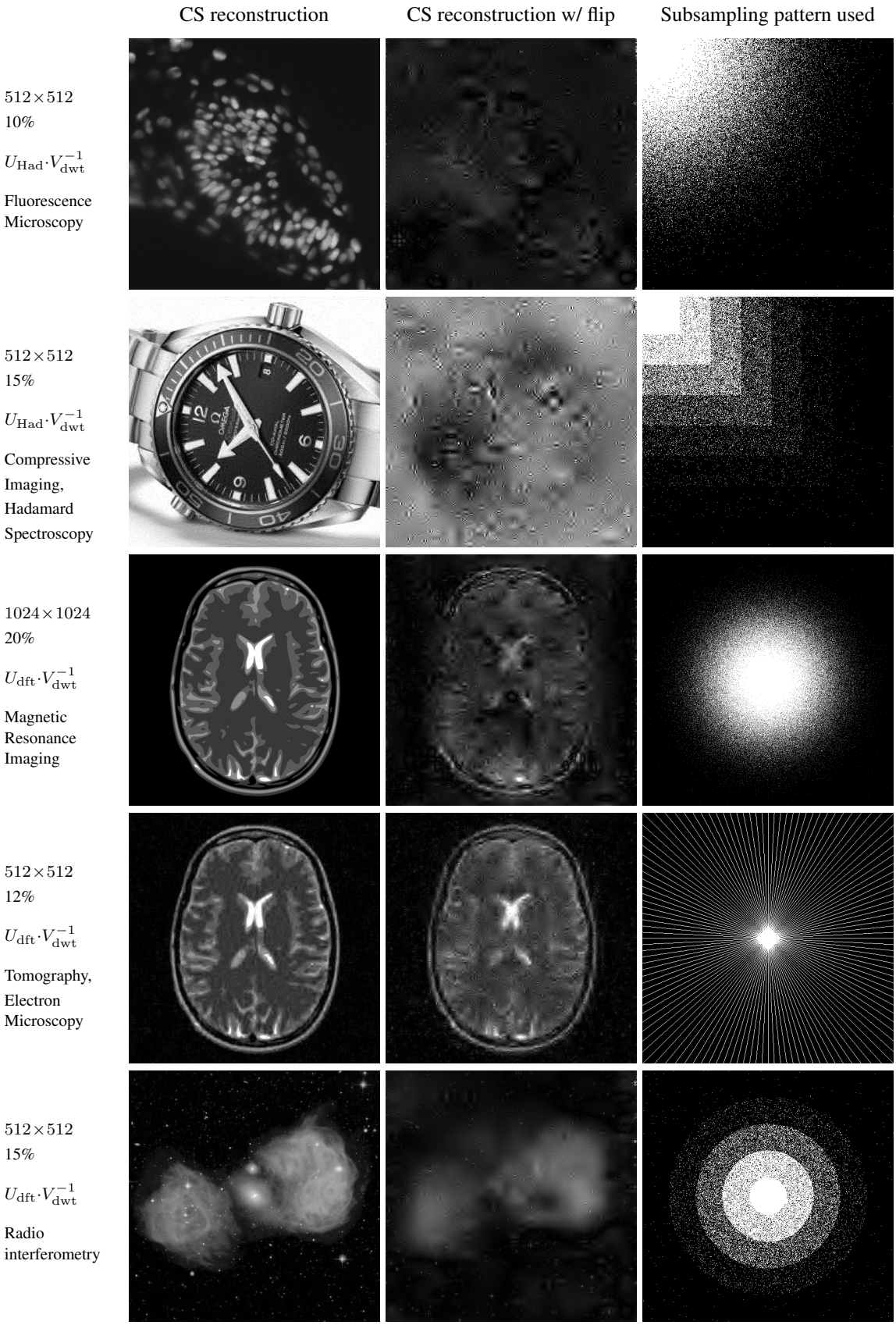


Figure 2: Reconstructions from subsampled coefficients from the direct wavelet coefficients (left column) and the flipped wavelet coefficients (middle column). The right column shows the subsampling map used. The percentage shown is the fraction of Fourier (DFT) or Walsh-Hadamard (WHT) coefficients that were sampled. The reconstruction basis was DB4 for the Fluorescence microscopy example, and DB6 for the rest.

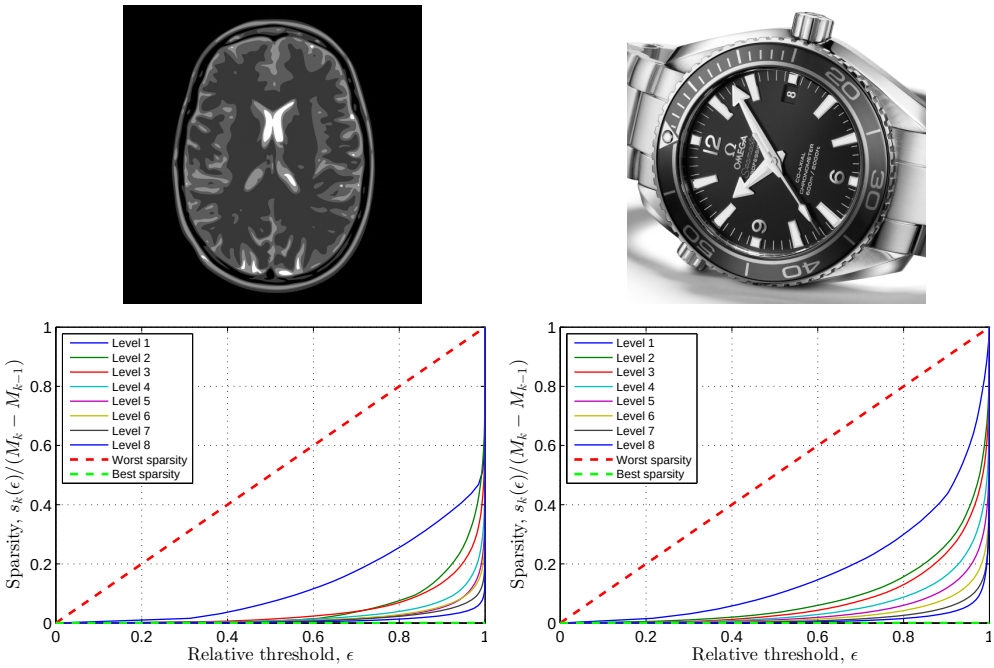


Figure 3: Relative sparsity of the Daubechies-8 wavelet coefficients of two images. Here the levels correspond to wavelet scales and $s_k(\epsilon)$ is given by (2.8). Each curve shows the relative sparsity at level k as a function of ϵ. The decreasing nature of the curves for increasing k confirms (2.9).

2.4 Real-world signals are asymptotically sparse

Given that structure is key, we now ask the question: what, if any, structure is characteristic of sparse signals in practice? Let us consider a wavelet basis $\{\varphi_n\}_{n\in\mathbb{N}}$. Recall that associated to such a basis, there is a natural decomposition of $\mathbb{N}$ into finite subsets according to different scales, i.e. $\mathbb{N} = \bigcup_{k\in\mathbb{N}}\{M_{k-1}+1,\ldots,M_k\}$, where $0 = M_0 < M_1 < M_2 < \ldots$ and $\{M_{k-1}+1,\ldots,M_k\}$ is the set of indices corresponding to the kth scale. Let $x \in l^2(\mathbb{N})$ be the coefficients of a function f in this basis. Suppose that $\epsilon \in (0,1]$ is given, and define

$$s_k = s_k(\epsilon) = \min\Big\{K : \Big\|\sum_{i=1}^{K} x_{\pi(i)}\varphi_{\pi(i)}\Big\| \geq \epsilon \Big\|\sum_{i=M_{k-1}+1}^{M_k} x_j\varphi_j\Big\|\Big\}, \tag{2.8}$$

where $\pi : \{1,\ldots,M_k - M_{k-1}\} \to \{M_{k-1}+1,\ldots,M_k\}$ is a bijection such that $|x_{\pi(i)}| \geq |x_{\pi(i+1)}|$ for $i = 1,\ldots,M_k - M_{k-1} - 1$. In order words, the quantity s_k is the effective sparsity of the wavelet coefficients of f at the kth scale.

Sparsity of f in a wavelet basis means that for a given maximal scale $r \in \mathbb{N}$, the ratio $s/M_r \ll 1$, where $M = M_r$ and $s = s_1 + \ldots + s_r$ is the total effective sparsity of f. The observation that typical images and signals are approximately sparse in wavelet bases is one of the key results in nonlinear approximation [25, 56]. However, such objects exhibit far more than sparsity alone. In fact, the ratios

$$s_k/(M_k - M_{k-1}) \to 0, \tag{2.9}$$

rapidly as $k \to \infty$, for every fixed $\epsilon \in (0,1]$. Thus typical signals and images have a distinct sparsity *structure*. They are much more sparse at fine scales (large k) than at coarse scales (small k). This is confirmed in Figure 3. This conclusion does not change fundamentally if one replaces wavelets by other related approximation systems, such as curvelets [11, 13], contourlets [26, 59] or shearlets [20, 21, 50].

3 New principles

Having shown their necessity, we now introduce the main new concepts of the paper: namely, asymptotic incoherence, asymptotic sparsity and multilevel sampling.

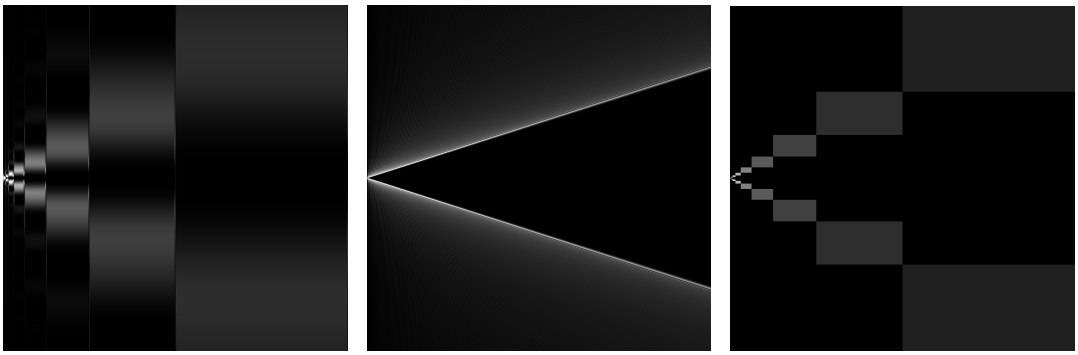


Figure 4: The absolute values of the matrix U in (2.5): (*left*): DB2 wavelets with Fourier sampling. (*middle*): Legendre polynomials with Fourier sampling. (*right*): The absolute values of $U_{\mathrm{Had}} V_{\mathrm{dwt}}^{-1}$, where U_{Had} is a Hadamard matrix and V_{dwt}^{-1} is the discrete Haar transform. Light regions correspond to large values and dark regions to small values.

3.1 Asymptotic incoherence

Recall from §2.2 that the case of Fourier sampling with wavelets as the sparsity basis is a standard example of a coherent problem. Similarly, Fourier sampling with Legendre polynomials is also coherent, as is the case of Hadamard sampling with wavelets. In Figure 4 we plot the absolute values of the entries of the matrix U for these three examples. As is evident, whilst U does indeed have large entries in all three case (since it is coherent), these are isolated to a leading submatrix (note that we enumerate over $\mathbb{Z}$ for the Fourier sampling basis and $\mathbb{N}$ for the wavelet/Legendre sparsity bases). As one moves away from this region the values get progressively smaller. That is, the matrix U is incoherent aside from a leading coherent submatrix. This motivates the following definition:

Definition 3.1 (Asymptotic incoherence)**.** *Let be $\{U_N\}$ be a sequence of isometries with $U_N \in \mathbb{C}^N$ or let $U \in l^2(\mathbb{N})$. Then*

(i) $\{U_N\}$ is asymptotically incoherent if $\mu(P_K^{\perp} U_N),\ \mu(U_N P_K^{\perp}) \to 0$, when $K \to \infty$, with $N/K = c$, for all $c \geq 1$.

(ii) U is asymptotically incoherent if $\mu(P_K^{\perp} U),\ \mu(U P_K^{\perp}) \to 0$, when $K \to \infty$.

Here P_K is the projection onto $\mathrm{span}\{e_j : j = 1, ..., K\}$, where $\{e_j\}$ is the canonical basis of either $\mathbb{C}^N$ or $l^2(\mathbb{N})$, and $P_K^{\perp}$ is its orthogonal complement.

In other words, U is asymptotically incoherent if the coherences of the matrices formed by replacing either the first K rows or columns of U are small. As it transpires, the Fourier/wavelets, Fourier/Legendre and Hadamard/wavelets problems are asymptotically incoherent. In particular, $\mu(P_K^{\perp} U),\ \mu(U P_K^{\perp}) = \mathcal{O}\left(K^{-1}\right)$ as $K \to \infty$ for the former (see §6).

3.2 Multi-level sampling

Asymptotic incoherence suggests a different subsampling strategy should be used instead of uniform random sampling. High coherence in the first few rows of U means that important information about the signal to be recovered may well be contained in its corresponding measurements. Hence to ensure good recovery we should fully sample these rows. Conversely, once outside of this region, when the coherence starts to decrease, we can begin to subsample. Let $N_1, N, m \in \mathbb{N}$ be given. This now leads us to consider an index set Ω of the form $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{1, \ldots, N_1\}$, and $\Omega_2 \subseteq \{N_1 + 1, \ldots, N\}$ is chosen uniformly at random with $|\Omega_2| = m$. We refer to this as a *two-level* sampling scheme. As we shall prove later, the amount of subsampling possible (i.e. the parameter m) in the region corresponding to Ω_2 will depend solely on the sparsity of the signal and coherence $\mu(P_{N_1}^{\perp} U)$.

The two-level scheme represents the simplest type of nonuniform density sampling. There is no reason, however, to restrict our attention to just two levels, full and subsampled. In general, we shall consider *multilevel* schemes, defined as follows:

Definition 3.2 (Multilevel random sampling)**.** *Let $r \in \mathbb{N}$, $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$ with $1 \leq N_1 < \ldots < N_r$, $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$, with $m_k \leq N_k - N_{k-1}$, $k = 1, \ldots, r$, and suppose that*

$$\Omega_k \subseteq \{N_{k-1} + 1, \ldots, N_k\}, \quad |\Omega_k| = m_k, \quad k = 1, \ldots, r,$$

are chosen uniformly at random, where $N_0 = 0$. We refer to the set

$$\Omega = \Omega_{\mathbf{N},\mathbf{m}} = \Omega_1 \cup \ldots \cup \Omega_r.$$

as an $(\mathbf{N}, \mathbf{m})$-multilevel sampling scheme.

The idea of sampling the low-order coefficients of an image differently goes back to the early days of compressed sensing. In particular, Donoho considers a two-level approach for recovering wavelet coefficients in his seminal paper [28], based on acquiring the coarse scale coefficients directly. This was later extended by Tsaig & Donoho to so-called 'multiscale compressed sensing' in [72], where distinct subbands were sensed separately. See also Romberg's work [63], and as well as Candès & Romberg [15].

Note that, although in part motivated by wavelets, our definition is completely general, as are the main theorems we present in §4 and §5. Moreover, and critically, we do not assume separation of the coefficients into distinct levels before sampling (as done above), which is often infeasible in practice (in particular, any application based on Fourier or Hadamard sampling). Note also that in MRI similar sampling strategies to what we introduce here are found in most implementations of compressed sensing [54, 55, 61, 62]. Additionally, a so-called "half-half" scheme (an example of a two-level strategy) was used by [67] in application of compressed sensing in fluorescence microscopy, albeit without theoretical recovery guarantees.

3.3 Asymptotic sparsity in levels

The flip test, the discussion in §2.4 and Figure 3 suggest that we need a different concept to sparsity. Given the structure of modern function systems such as wavelets and their generalizations, we propose the notion of sparsity in levels:

Definition 3.3 (Sparsity in levels)**.** *Let x be an element of either $\mathbb{C}^N$ or $l^2(\mathbb{N})$. For $r \in \mathbb{N}$ let $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$ with $1 \leq M_1 < \ldots < M_r$ and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, with $s_k \leq M_k - M_{k-1}$, $k = 1, \ldots, r$, where $M_0 = 0$. We say that x is $(\mathbf{s}, \mathbf{M})$-sparse if, for each $k = 1, \ldots, r$,*

$$\Delta_k := \mathrm{supp}(x) \cap \{M_{k-1} + 1, \ldots, M_k\},$$

satisfies $|\Delta_k| \leq s_k$. We denote the set of $(\mathbf{s}, \mathbf{M})$-sparse vectors by $\Sigma_{\mathbf{s},\mathbf{M}}$.

Definition 3.4 ($(\mathbf{s}, \mathbf{M})$-term approximation)**.** *Let $f = \sum_{j \in \mathbb{N}} x_j \varphi_j$, where $x = (x_j)_{j \in \mathbb{N}}$ is an element of either $\mathbb{C}^N$ or $l^2(\mathbb{N})$. We say that f is $(\mathbf{s}, \mathbf{M})$-compressible with respect to $\{\varphi_j\}_{j \in \mathbb{N}}$ if $\sigma_{\mathbf{s},\mathbf{M}}(f)$ is small, where*

$$\sigma_{\mathbf{s},\mathbf{M}}(f) = \min_{\eta \in \Sigma_{\mathbf{s},\mathbf{M}}} \|x - \eta\|_{l^1}. \tag{3.1}$$

Typically, it is the case that $s_k/(M_k - M_{k-1}) \to 0$ as $k \to \infty$, in which case we say that x is *asymptotically sparse in levels*. However, our theory does not explicitly require such decay. As we shall see next, vectors x that are (asymptotically) sparse in levels are ideally suited to multilevel sampling schemes. Roughly speaking, under the appropriate conditions, the number of measurements m_k required in each sampling band Ω_k is determined by the sparsity of x in the corresponding sparsity band Δ_k and the asymptotic coherence.

4 Main theorems I: the finite-dimensional case

We now present the main theorems in the finite-dimensional setting. In §5 we address the infinite-dimensional case. To avoid pathological examples we will assume throughout that the total sparsity $s = s_1 + \ldots + s_r \geq 3$. This is simply to ensure that $\log(s) \geq 1$, which is convenient in the proofs.

4.1 Two-level sampling schemes

We commence with the case of two-level sampling schemes. Recall that in practice, signals are never exactly sparse (or sparse in levels), and their measurements are always contaminated by noise. Let $f = \sum_j x_j \varphi_j$ be a fixed signal, and write

$$y = P_\Omega \hat{f} + z = P_\Omega U x + z,$$

for its noisy measurements, where $z \in \mathrm{ran}(P_\Omega)$ is a noise vector satisfying $\|z\| \leq \delta$ for some $\delta \geq 0$. If δ is known, we now consider the following problem:

$$\min_{\eta \in \mathbb{C}^N} \|\eta\|_{l^1} \text{ subject to } \|P_\Omega U \eta - y\| \leq \delta. \tag{4.1}$$

Our aim in this setting is to recover x up to an error proportional to δ and the best $(\mathbf{s}, \mathbf{M})$-term approximation error $\sigma_{\mathbf{s},\mathbf{M}}(x)$. Before stating our theorem, it is useful to make the following definition: $\mu_K = \mu(P_K^\perp U)$, for $K \in \mathbb{N}$. We now have the following:

Theorem 4.1. *Let $U \in \mathbb{C}^{N \times N}$ be an isometry and $x \in \mathbb{C}^N$. Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a two-level sampling scheme, where $\mathbf{N} = (N_1, N_2)$, $N_2 = N$, and $\mathbf{m} = (N_1, m_2)$. Let $(\mathbf{s}, \mathbf{M})$, where $\mathbf{M} = (M_1, M_2) \in \mathbb{N}^2$, $M_1 < M_2$, $M_2 = N$, and $\mathbf{s} = (M_1, s_2) \in \mathbb{N}^2$, $s_2 \leq M_2 - M_1$, be any pair such that the following holds:*

(i) we have

$$\|P_{N_1}^\perp U P_{M_1}\| \leq \frac{\gamma}{\sqrt{M_1}} \tag{4.2}$$

and $\gamma \leq s_2 \sqrt{\mu_{N_1}}$ for some $\gamma \in (0, 2/5]$;

(ii) for $\epsilon \in (0, e^{-1}]$, let

$$m_2 \gtrsim (N - N_1) \cdot \log(\epsilon^{-1}) \cdot \mu_{N_1} \cdot s_2 \cdot \log(N).$$

Suppose that $\xi \in l^1(\mathbb{N})$ is a minimizer of (4.1). Then, with probability exceeding $1 - s\epsilon$, we have

$$\|\xi - x\| \leq C \cdot \left(\delta \cdot \sqrt{K} \cdot \left(1 + L \cdot \sqrt{s}\right) + \sigma_{\mathbf{s},\mathbf{M}}(f) \right), \tag{4.3}$$

for some constant C, where $\sigma_{\mathbf{s},\mathbf{M}}(f)$ is as in (3.1), $L = 1 + \frac{\sqrt{\log_2(6\epsilon^{-1})}}{\log_2(4KM\sqrt{s})}$ and $K = (N_2 - N_1)/m_2$. If $m_2 = N - N_1$ then this holds with probability 1.

Let us now interpret Theorem 4.1, and in particular, demonstrates how it overcomes the coherence barrier. We note the following:

(i) The condition $\|P_{N_1}^\perp U P_{M_1}\| \leq \frac{2}{5\sqrt{M_1}}$ (which is always satisfied for some N_1) implies that fully sampling the first N_1 measurements allows one to recover the first M_1 coefficients of f.

(ii) To recover the remaining s_2 coefficients we require, up to log factors, an additional

$$m_2 \gtrsim (N - N_1) \cdot \mu_{N_1} \cdot s_2,$$

measurements, taken randomly from the range $M_1 + 1, \ldots, M_2$. In particular, if N_1 is a fixed fraction of N, and if $\mu_{N_1} = \mathcal{O}\left(N_1^{-1}\right)$, such as for wavelets with Fourier measurements (Theorem 6.1), then one requires only $m_2 \gtrsim s_2$ additional measurements to recover the sparse part of the signal.

Thus, in the case where x is asymptotically sparse, we require a fixed number N_1 measurements to recover the nonsparse part of x, and then a number m_2 depending on s_2 and the asymptotic coherence μ_{N_1} to recover the sparse part.

Remark 4.1 It is not necessary to know the sparsity structure, i.e. the values $\mathbf{s}$ and $\mathbf{M}$, of the image f in order to implement the two-level sampling technique (the same also applies to the multilevel technique discussed in the next section). Given a two-level scheme $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$, Theorem 4.1 demonstrates that f will be recovered exactly up to an error on the order of $\sigma_{\mathbf{s},\mathbf{M}}(f)$, where $\mathbf{s}$ and $\mathbf{M}$ are determined implicitly by $\mathbf{N}$, $\mathbf{m}$ and the conditions (i) and (ii) of the theorem. Of course, some *a priori* knowledge of $\mathbf{s}$ and $\mathbf{M}$ will greatly assist in selecting the parameters $\mathbf{N}$ and $\mathbf{m}$ so as to get the best recovery results. However, this is not necessary for implementation.

4.2 Multilevel sampling schemes

We now consider the case of multilevel sampling schemes. Before presenting this case, we need several definitions. The first is key concept in this paper: namely, *the local coherence*.

Definition 4.2 (Local coherence)**.** *Let U be an isometry of either $\mathbb{C}^N$ or $l^2(\mathbb{N})$. If $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$ and $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$ with $1 \leq N_1 < \ldots N_r$ and $1 \leq M_1 < \ldots < M_r$ the $(k,l)^{\text{th}}$ local coherence of U with respect to $\mathbf{N}$ and $\mathbf{M}$ is given by*

$$\mu_{\mathbf{N},\mathbf{M}}(k,l) = \sqrt{\mu(P_{N_k}^{N_{k-1}} U P_{M_l}^{M_{l-1}}) \cdot \mu(P_{N_k}^{N_{k-1}} U)}, \quad k,l = 1, \ldots, r,$$

where $N_0 = M_0 = 0$ and P_b^a denotes the projection matrix corresponding to indices $\{a+1, \ldots, b\}$. In the case where $U \in \mathcal{B}(l^2(\mathbb{N}))$ (i.e. U belongs to the space of bounded operators on $l^2(\mathbb{N})$), we also define

$$\mu_{\mathbf{N},\mathbf{M}}(k,\infty) = \sqrt{\mu(P_{N_k}^{N_{k-1}} U P_{M_{r-1}}^{\perp}) \cdot \mu(P_{N_k}^{N_{k-1}} U)}, \quad k = 1, \ldots, r.$$

Besides the local sparsities s_k, we shall also require the notion of a relative sparsity:

Definition 4.3 (Relative sparsity)**.** *Let U be an isometry of either $\mathbb{C}^N$ or $l^2(\mathbb{N})$. For $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$, $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$ with $1 \leq N_1 < \ldots < N_r$ and $1 \leq M_1 < \ldots < M_r$, $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $1 \leq k \leq r$, the kth relative sparsity is given by*

$$S_k = S_k(\mathbf{N}, \mathbf{M}, \mathbf{s}) = \max_{\eta \in \Theta} \|P_{N_k}^{N_{k-1}} U \eta\|^2,$$

where $N_0 = M_0 = 0$ and Θ is the set

$$\Theta = \{\eta : \|\eta\|_{l^\infty} \leq 1, |\text{supp}(P_{M_l}^{M_{l-1}} \eta)| = s_l,\, l = 1, \ldots, r\}.$$

We can now present our main theorem:

Theorem 4.4. *Let $U \in \mathbb{C}^{N \times N}$ be an isometry and $x \in \mathbb{C}^N$. Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a multilevel sampling scheme, where $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$, $N_r = N$, and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Let $(\mathbf{s}, \mathbf{M})$, where $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$, $M_r = N$, and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, be any pair such that the following holds: for $\epsilon \in (0, e^{-1}]$ and $1 \leq k \leq r$,*

$$1 \gtrsim \frac{N_k - N_{k-1}}{m_k} \cdot \log(\epsilon^{-1}) \cdot \left(\sum_{l=1}^{r} \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot s_l \right) \cdot \log(N), \tag{4.4}$$

where $m_k \gtrsim \hat{m}_k \cdot \log(\epsilon^{-1}) \cdot \log(N)$, and $\hat{m}_k$ is such that

$$1 \gtrsim \sum_{k=1}^{r} \left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1 \right) \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot \tilde{s}_k, \tag{4.5}$$

for all $l = 1, \ldots, r$ and all $\tilde{s}_1, \ldots, \tilde{s}_r \in (0, \infty)$ satisfying

$$\tilde{s}_1 + \ldots + \tilde{s}_r \leq s_1 + \ldots + s_r, \qquad \tilde{s}_k \leq S_k(\mathbf{N}, \mathbf{M}, \mathbf{s}).$$

Suppose that $\xi \in \mathbb{C}^N$ is a minimizer of (4.1). Then, with probability exceeding $1 - s\epsilon$, where $s = s_1 + \ldots + s_r$, we have that

$$\|\xi - x\| \leq C \cdot \left(\delta \cdot \sqrt{K} \cdot \left(1 + L \cdot \sqrt{s}\right) + \sigma_{\mathbf{s},\mathbf{M}}(f) \right),$$

for some constant C, where $\sigma_{\mathbf{s},\mathbf{M}}(f)$ is as in (3.1), $L = 1 + \frac{\sqrt{\log_2(6\epsilon^{-1})}}{\log_2(4KM\sqrt{s})}$ and $K = \max_{1 \leq k \leq r}\{(N_k - N_{k-1})/m_k\}$. If $m_k = N_k - N_{k-1}$, $1 \leq k \leq r$, then this holds with probability 1.

The key component of this theorem are the bounds (4.4) and (4.5). Whereas the standard compressed sensing estimate (2.2) relates the total number of samples m to the global coherence and the global sparsity, these bounds now relate the local sampling m_k to the local coherences $\mu_{\mathbf{N},\mathbf{M}}(k,l)$ and local and relative

sparsities s_k and S_k. In particular, by relating these local quantities this theorem conforms with the conclusions of the flip test in §2.3: namely, the optimal sampling strategy must depend on the signal structure, and this is exactly what is advocated in (4.4) and (4.5).

On the face of it, the bounds (4.4) and (4.5) may appear somewhat complicated, not least because they involve the relative sparsities S_k. As we next show, however, they are indeed sharp in the sense that they reduce to the correct information-theoretic limits in several important cases. Furthermore, in the important case of wavelet sparsity with Fourier sampling, they can be used to provide near-optimal recovery guarantees. We discuss this in §6. Note, however, that to do this it is first necessary to generalize Theorem 4.4 to the infinite-dimensional setting, which we do in §5.

4.2.1 Sharpness of the estimates – the block-diagonal case

Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a multilevel sampling scheme, where $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Let $(\mathbf{s}, \mathbf{M})$, where $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$, and suppose for simplicity that $\mathbf{M} = \mathbf{N}$. Consider the block-diagonal matrix

$$\mathbb{C}^{N\times N} \ni A = \bigoplus_{k=1}^{r} A_k, \quad A_k \in \mathbb{C}^{(N_k - N_{k-1})\times(N_k - N_{k-1})}, \quad A_k^* A_k = I,$$

where $N_0 = 0$. Note that in this setting we have $S_k = s_k$, $\mu_{\mathbf{N},\mathbf{M}}(k,l) = 0$, $k \neq l$. Also, since $\mu(\mathbf{N},\mathbf{M})(k,k) = \mu(A_k)$, equations (4.4) and (4.5) reduce to

$$1 \gtrsim \frac{N_k - N_{k-1}}{m_k} \cdot \log(\epsilon^{-1}) \cdot \mu(A_k) \cdot s_k \cdot \log N, \quad 1 \gtrsim \left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1 \right) \cdot \mu(A_k) \cdot s_k.$$

In particular, it suffices to take

$$m_k \gtrsim (N_k - N_{k-1}) \cdot \log(\epsilon^{-1}) \cdot \mu(A_k) \cdot s_k \cdot \log N, \quad 1 \leq k \leq r. \tag{4.6}$$

This is exactly as one expects: the number of measurements in the kth level depends on the size of the level multiplied by the asymptotic incoherence and the sparsity in that level. Note that this result recovers the standard one-level results in finite dimensions [1, 14] up to the $1 - s\epsilon$ bound on the probability. In particular, the typical bound would be $1 - \epsilon$. The question as to whether or not this s can be removed in the multilevel setting is open, although such a result would be more of a cosmetic improvement.

4.2.2 Sharpness of the estimates – the non-block diagonal case

The previous argument demonstrated that Theorem 4.4 is sharp, up to the probability term, in the sense that it reduces to the usual estimate (4.6) for block-diagonal matrices. A key step in showing this is noting that the quantities S_k reduce to the sparsities s_k in the block-diagonal case. Unfortunately, this is not true in the general setting. Note that one has the upper bound $S_k \leq s = s_1 + \ldots + s_r$. However in general there is usually *interference* between different sparsity levels, which means that S_k need not have anything to do with s_k, or can indeed be proportional to the total sparsity s. This may seem an undesirable aspect of the theorems, since S_k may be significantly larger than s_k, and thus the estimate on the number of measurements m_k required in the kth level may also be much larger than the corresponding sparsity s_k. Could it therefore be that the S_ks are an unfortunate artefact of the proof? As we now show by example, this is not the case.

To do this, we consider the following setting. Let $N = rn$ for some $n \in \mathbb{N}$ and $\mathbf{N} = \mathbf{M} = (n, 2n, \ldots, rn)$. Let $W \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{r\times r}$ be isometries and consider the matrix $A = V \otimes W$, where $\otimes$ is the usual Kronecker product. Note that $A \in \mathbb{C}^{N\times N}$ is also an isometry. Now suppose that $x = (x_1, \ldots, x_r) \in \mathbb{C}^N$ is an $(\mathbf{s}, \mathbf{M})$-sparse vector, where each $x_k \in \mathbb{C}^n$ is s_k-sparse. Then

$$Ax = y, \quad y = (y_1, \ldots, y_r),\ y_k = W z_k,\ z_k = \sum_{l=1}^{r} v_{kl} x_l.$$

Hence the problem of recovering x from measurements y with an $(\mathbf{N}, \mathbf{m})$-multilevel strategy decouples into r problems of recovering the vector z_k from the measurements $y_k = W z_k$, $k = 1, \ldots, r$. Let $\hat{s}_k$ denote the sparsity of z_k. Since the coherence provides an information-theoretic limit [14], one requires at least

$$m_k \gtrsim n \cdot \mu(W) \cdot \hat{s}_k \cdot \log n, \quad 1 \leq k \leq r. \tag{4.7}$$

measurements at level k in order to recover each z_k, and therefore recover x, regardless of the reconstruction method used. We now consider two examples of this setup:

Example 4.1 Let $\pi : \{1, \ldots, r\} \to \{1, \ldots, r\}$ be a permutation and let V be the matrix with entries $v_{kl} = \delta_{l,\pi(k)}$. Since $z_k = x_{\pi(k)}$ in this case, the lower bound (4.7) reads

$$m_k \gtrsim n \cdot \mu(W) \cdot s_{\pi(k)} \cdot \log n, \quad 1 \leq k \leq r. \tag{4.8}$$

Now consider Theorem 4.4 for this matrix. First, we note that $S_k = s_{\pi(k)}$. In particular, S_k is completely unrelated to s_k. Substituting this into Theorem 4.4 and noting that $\mu_{\mathbf{N},\mathbf{M}}(k,l) = \mu(W)\delta_{l,\pi(k)}$ in this case, we arrive at the condition $m_k \gtrsim n \cdot \mu(W) \cdot s_{\pi(k)} \cdot \big(\log(\epsilon^{-1}) + 1\big) \cdot \log(nr)$, which is equivalent to (4.8).

Example 4.2 Now suppose that V is the $r \times r$ DFT matrix. Suppose also that $s \leq n/r$ and that the x_k's have disjoint support sets, i.e. $\mathrm{supp}(x_k) \cap \mathrm{supp}(x_l) = \emptyset$, $k \neq l$. Then by construction, each z_k is s-sparse, and therefore the lower bound (4.7) reads $m_k \gtrsim n \cdot \mu(W) \cdot s \cdot \log n$, for $1 \leq k \leq r$. After a short argument, one finds that $s/r \leq S_k \leq s$ in this case. Hence, S_k is typically much larger than s_k. Moreover, after noting that $\mu_{\mathbf{N},\mathbf{M}}(k,l) = \frac{1}{r}\mu(W)$, we find that Theorem 4.4 gives the condition $m_k \gtrsim n \cdot \mu(W) \cdot s \cdot \big(\log(\epsilon^{-1}) + 1\big) \cdot \log(nr)$. Thus, Theorem 4.4 obtains the lower bound in this case as well.

4.2.3 Sparsity leads to pessimistic reconstruction guarantees

Recall that the flip test demonstrates that any sparsity-based theory of compressed sensing does not describe the reconstructions seen in practice. To conclude this section, we now use the block-diagonal case to further emphasize the need for theorems that go beyond sparsity, such as Theorems 4.1 and 4.4. To see this, consider the block-diagonal matrix

$$U = \bigoplus_{k=1}^{r} U_r, \qquad U_k \in \mathbb{C}^{(N_k - N_{k-1}) \times (N_k - N_{k-1})},$$

where each U_k is perfectly incoherent, i.e. $\mu(U_k) = (N_k - N_{k-1})^{-1}$, and suppose we take m_k measurements within each block U_k. Let $x \in \mathbb{C}^N$ be the signal we wish to recover, where $N = N_r$. The question is, how many samples $m = m_1 + \ldots + m_r$ do we require?

Suppose we assume that x is s-sparse, where $s \leq \min_{k=1,\ldots,r}\{N_k - N_{k-1}\}$. Given no further information about the sparsity structure, it is necessary to take $m_k \gtrsim s\log(N)$ measurements in each block, giving $m \gtrsim rs\log(N)$ in total. However, suppose now that x is known to be s_k-sparse within each level, i.e. $|\mathrm{supp}(x) \cap \{N_{k-1} + 1, \ldots, N_k\}| = s_k$. Then we now require only $m_k \gtrsim s_k \log(N)$, and therefore $m \gtrsim s\log(N)$ total measurements. Thus, structured sparsity leads to a significant saving by a factor of r in the total number of measurements required.

5 Main theorems II: the infinite-dimensional case

Finite-dimensional compressed sensing is suitable in many cases. However, there are some important problems where this framework can lead to significant problems, since the underlying problem is continuous/analog. Discretization of the problem in order to produce a finite-dimensional, vector-space model can lead to substantial errors [1, 7, 18, 66], due to the phenomenon of model mismatch such as the inverse or the wavelet crimes.

To address this issue, a theory of compressed sensing in infinite dimensions was introduced by Adcock & Hansen in [1], based on a new approach to classical sampling known as *generalized sampling* [2, 3, 4, 5]. We describe this theory next. Note that this infinite-dimensional compressed sensing model has also been advocated and implemented in MRI by Guerquin–Kern, Häberlin, Pruessmann & Unser [40]. Furthermore, we shall see in §6 that the infinite-dimensional analysis is crucial in order to obtain accurate recovery estimates in the Fourier/wavelets case.

5.1 Infinite-dimensional compressed sensing

Let us now describe the framework of [1] in more detail. Suppose that $\mathcal{H}$ is a separable Hilbert space over $\mathbb{C}$, and let $\{\psi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis on $\mathcal{H}$ (the sampling basis). Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an orthonormal system

in $\mathcal{H}$ (the sparsity system), and suppose that

$$U = (u_{ij})_{i,j\in\mathbb{N}}, \qquad u_{ij} = \langle \varphi_j, \psi_i \rangle, \tag{5.1}$$

is an infinite matrix. We may consider U as an element of $\mathcal{B}(l^2(\mathbb{N}))$; the space of bounded operators on $l^2(\mathbb{N})$ (we will make no distinction between bounded operators on sequence spaces and infinite matrices). As in the finite-dimensional case, U is an isometry, and we may define its coherence $\mu(U) \in (0,1]$ analogously to (2.1). We say that an element $f \in \mathcal{H}$ is (s,M)-sparse with respect to $\{\varphi_j\}_{j\in\mathbb{N}}$, where $s, M \in \mathbb{N}$, $s \leq M$, if the following holds: $f = \sum_{j\in\mathbb{N}} x_j \varphi_j$, $\mathrm{supp}(x) = \{j : x_j \neq 0\} \subseteq \{1,\ldots,M\}$, $|\mathrm{supp}(x)| \leq s$. Setting $\Sigma_{s,M} = \left\{x \in l^2(\mathbb{N}) : x \text{ is } (s,M)\text{-sparse}\right\}$, we define $\sigma_{s,M}(f) = \min_{\eta\in\Sigma_{s,M}} \|x-\eta\|_{l^1}$, with $f = \sum_{j\in\mathbb{N}} x_j\varphi_j$, $x = (x_j)_{j\in\mathbb{N}} \in l^1(\mathbb{N})$, and say that f is (s,M)-compressible with respect to $\{\varphi_j\}_{j\in\mathbb{N}}$ if $\sigma_{s,M}(f)$ is small. Whenever f is (s,M)-sparse or compressible, we seek to recover it from a small number of the measurements $\hat{f}_j = \langle f, \psi_j \rangle$, $j \in \mathbb{N}$. To do this, we introduce a second parameter $N \in \mathbb{N}$, and let Ω be a randomly-chosen subset of indices $1,\ldots,N$ of size m. Unlike in finite dimensions, we now consider two cases. Suppose first that $P_M^\perp x = 0$, i.e. x has no tail. Then we solve

$$\inf_{\eta\in l^1(\mathbb{N})} \|\eta\|_{l^1} \text{ subject to } \|P_\Omega U P_M \eta - y\| \leq \delta, \tag{5.2}$$

where $y = P_\Omega \hat{f} + z$, $\hat{f} = (\hat{f}_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N})$, $z \in \mathrm{ran}(P_\Omega)$ is a noise vector satisfying $\|z\| \leq \delta$, and P_Ω is the projection operator corresponding to the index set Ω. In [1] it was proved that any solution to (5.2) recovers f exactly up to an error determined by $\sigma_{s,M}(f)$, provided N and m satisfy the so-called *weak balancing property* with respect to M and s (see Definition 5.1, as well as Remark 5.1 for a discussion), and provided

$$m \gtrsim \mu(U) \cdot N \cdot s \cdot \left(1 + \log(\epsilon^{-1})\right) \cdot \log\left(m^{-1} M N \sqrt{s}\right). \tag{5.3}$$

As in the finite-dimensional case, which turns out to be a corollary of this result, we find that m is on the order of the sparsity s whenever $\mu(U)$ is sufficiently small.

In practice, the condition $P_M^\perp x = 0$ is unrealistic. In the more general case, $P_M^\perp x \neq 0$, we solve the following problem:

$$\inf_{\eta\in l^1(\mathbb{N})} \|\eta\|_{l^1} \text{ subject to } \|P_\Omega U \eta - y\| \leq \delta. \tag{5.4}$$

In [1] it was shown that any solution of (5.4) recovers f exactly up to an error determined by $\sigma_{s,M}(f)$, provided N and m satisfy the so-called *strong balancing property* with respect to M and s (see Definition 5.1), and provided a bound similar to (5.3) holds, where the M is replaced by a slightly larger constant (we give the details in the next section in the more general setting of multilevel sampling). Note that (5.4) cannot be solve numerically, since it is infinite-dimensional. Therefore in practice we replace (5.4) by

$$\inf_{\eta\in l^1(\mathbb{N})} \|\eta\|_{l^1} \text{ subject to } \|P_\Omega U P_R \eta - y\| \leq \delta, \tag{5.5}$$

where R is taken sufficiently large. See [1] for the details.

5.2 Main theorems

We first require the definition of the so-called *balancing property* [1]:

Definition 5.1 (Balancing property)**.** *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ be an isometry. Then $N \in \mathbb{N}$ and $K \geq 1$ satisfy the weak balancing property with respect to U, $M \in \mathbb{N}$ and $s \in \mathbb{N}$ if*

$$\|P_M U^* P_N U P_M - P_M\|_{l^\infty \to l^\infty} \leq \frac{1}{8}\left(\log_2^{1/2}\left(4\sqrt{s}KM\right)\right)^{-1}, \tag{5.6}$$

where $\|\cdot\|_{l^\infty\to l^\infty}$ is the norm on $\mathcal{B}(l^\infty(\mathbb{N}))$. We say that N and K satisfy the strong balancing property with respect to U, M and s if (5.6) holds, as well as

$$\|P_M^\perp U^* P_N U P_M\|_{l^\infty\to l^\infty} \leq \frac{1}{8}. \tag{5.7}$$

As in the previous section, we commence with the two-level case. Furthermore, to illustrate the differences between the weak/strong balancing property, we first consider the setting of (5.2):

Theorem 5.2. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ be an isometry and $x \in l^1(\mathbb{N})$. Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a two-level sampling scheme, where $\mathbf{N} = (N_1, N_2)$ and $\mathbf{m} = (N_1, m_2)$. Let $(\mathbf{s}, \mathbf{M})$, where $\mathbf{M} = (M_1, M_2) \in \mathbb{N}^2$, $M_1 < M_2$, and $\mathbf{s} = (M_1, s_2) \in \mathbb{N}^2$, be any pair such that the following holds:*

(i) we have $\|P_{N_1}^{\perp} U P_{M_1}\| \leq \frac{\gamma}{\sqrt{M_1}}$ and $\gamma \leq s_2 \sqrt{\mu_{N_1}}$ for some $\gamma \in (0, 2/5]$;

(ii) the parameters

$$N = N_2, \qquad K = (N_2 - N_1)/m_2$$

satisfy the weak balancing property with respect to U, $M := M_2$ and $s := M_1 + s_2$;

(iii) for $\epsilon \in (0, e^{-1}]$, let

$$m_2 \gtrsim (N - N_1) \cdot \log(\epsilon^{-1}) \cdot \mu_{N_1} \cdot s_2 \cdot \log\left(KM\sqrt{s}\right).$$

Suppose that $P_{M_2}^{\perp} x = 0$ and let $\xi \in l^1(\mathbb{N})$ be a minimizer of (5.2). Then, with probability exceeding $1 - s\epsilon$, we have

$$\|\xi - x\| \leq C \cdot \left(\delta \cdot \sqrt{K} \cdot \left(1 + L \cdot \sqrt{s}\right) + \sigma_{\mathbf{s},\mathbf{M}}(f)\right), \tag{5.8}$$

for some constant C, where $\sigma_{\mathbf{s},\mathbf{M}}(f)$ is as in (3.1), and $L = 1 + \frac{\sqrt{\log_2(6\epsilon^{-1})}}{\log_2(4KM\sqrt{s})}$. If $m_2 = N - N_1$ then this holds with probability 1.

We next state a result for multilevel sampling in the more general setting of (5.4). For this, we require the following notation:

$$\tilde{M} = \min\{i \in \mathbb{N} : \max_{k \geq i} \|P_N U e_k\| \leq 1/(32K\sqrt{s})\},$$

where N, s and K are as defined below.

Theorem 5.3. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ be an isometry and $x \in l^1(\mathbb{N})$. Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a multilevel sampling scheme, where $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Let $(\mathbf{s}, \mathbf{M})$, where $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$, $M_1 < \ldots < M_r$, and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, be any pair such that the following holds:*

(i) the parameters

$$N = N_r, \quad K = \max_{k=1,\ldots,r} \left\{\frac{N_k - N_{k-1}}{m_k}\right\},$$

satisfy the strong balancing property with respect to U, $M := M_r$ and $s := s_1 + \ldots + s_r$;

(ii) for $\epsilon \in (0, e^{-1}]$ and $1 \leq k \leq r$,

$$1 \gtrsim \frac{N_k - N_{k-1}}{m_k} \cdot \log(\epsilon^{-1}) \cdot \left(\sum_{l=1}^{r} \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot s_l\right) \cdot \log\left(K\tilde{M}\sqrt{s}\right),$$

(with $\mu_{\mathbf{N},\mathbf{M}}(k,r)$ replaced by $\mu_{\mathbf{N},\mathbf{M}}(k,\infty)$) and $m_k \gtrsim \hat{m}_k \cdot \log(\epsilon^{-1}) \cdot \log\left(K\tilde{M}\sqrt{s}\right)$, where $\hat{m}_k$ satisfies (4.5).

Suppose that $\xi \in l^1(\mathbb{N})$ is a minimizer of (4.1). Then, with probability exceeding $1 - s\epsilon$,

$$\|\xi - x\| \leq C \cdot \left(\delta \cdot \sqrt{K} \cdot \left(1 + L \cdot \sqrt{s}\right) + \sigma_{\mathbf{s},\mathbf{M}}(f)\right),$$

for some constant C, where $\sigma_{\mathbf{s},\mathbf{M}}(f)$ is as in (3.1), and $L = C \cdot \left(1 + \frac{\sqrt{\log_2(6\epsilon^{-1})}}{\log_2(4KM\sqrt{s})}\right)$. If $m_k = N_k - N_{k-1}$ for $1 \leq k \leq r$ then this holds with probability 1.

This theorem removes the condition in Theorem 5.2 that x has zero tail. Note that the price to pay is the $\tilde{M}$ in the logarithmic term rather than M ($\tilde{M} \geq M$ because of the balancing property). Observe that $\tilde{M}$ is finite, and in the case of Fourier sampling with wavelets, we have that $\tilde{M} = \mathcal{O}(KN)$ (see §6). Note that Theorem 5.2 has a strong form analogous to Theorem 5.3 which removes the tail condition. The only difference is the requirement of the strong, as opposed to the weak, balancing property, and the replacement of M by $\tilde{M}$ in the log factor. Similarly, Theorem 5.3 has a weak form involving a tail condition. For succinctness we do not state these.

Remark 5.1 The balancing property is the main difference between the finite- and infinite-dimensional theorems. Its role is to ensure that the truncated matrix $P_N U P_M$ is close to an isometry. In reconstruction problems, the presence of an isometry ensures stability in the mapping between measurements and coefficients [2], which explains the need for a such a property in our theorems. As explained in [1], without the balancing property the lack of stability in the underlying mapping leads to numerically useless reconstructions. Note that the balancing property is usually not satisfied for $N = M$, and this choice typically leads to numerical instability. In general, one requires $N > M$ for the balancing property to hold. However, there is always a finite N for which it is satisfied, since the infinite matrix U is an isometry. For details we refer to [1]. We will provide specific estimates in §6 for required magnitude of N for the case of Fourier sampling with wavelet sparsity.

6 Recovery of wavelet coefficients from Fourier samples

Fourier sampling with wavelets as the sparsity system is a fundamentally important reconstruction problem in compressed sensing, with numerous applications ranging from medical imaging (e.g. MRI, X-ray CT via the Fourier slice theorem) to seismology and interferometry. We consider only the one-dimensional case for simplicity, since the extension to higher dimensions is conceptually straightforward. The incoherence properties can be described as follows.

Theorem 6.1. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ be the matrix corresponding to the Fourier/wavelets system described in §7.4. Then $\mu(U) \geq \omega|\hat{\Phi}(0)|^2$, where ω is the sampling density and Φ is the corresponding scaling function. Furthermore, $\mu(P_N^{\perp}U), \mu(UP_N^{\perp}) = \mathcal{O}\left(N^{-1}\right)$ as $N \to \infty$.*

Thus, Fourier sampling with wavelet sparsity is indeed globally coherent, yet asymptotically incoherent. This result holds for essentially any wavelet basis in one dimension (see [46] for the multidimensional case). To recover wavelet coefficients, we now apply a multilevel sampling strategy. This raises the question: how do we design this strategy, and how many measurements are required? If the levels $\mathbf{M} = (M_1, \ldots, M_r)$ correspond to the wavelet scales, and $\mathbf{s} = (s_1, \ldots, s_r)$ to the sparsities within them, then the best one could hope to achieve is that the number of measurements m_k in the kth sampling level is proportional to the sparsity s_k in the corresponding sparsity level. Our main theorem below shows that multilevel sampling can achieve this, up to an exponentially-localized factor and the usual log terms.

Theorem 6.2. *Consider an orthonormal basis of compactly supported wavelets with a multiresolution analysis (MRA). Let Φ and Ψ denote the scaling function and mother wavelet respectively, and let $\alpha \geq 1$ be such that*

$$\left|\hat{\Phi}(\xi)\right| \leq \frac{C}{(1+|\xi|)^{\alpha}}, \quad \left|\hat{\Psi}(\xi)\right| \leq \frac{C}{(1+|\xi|)^{\alpha}}, \qquad \xi \in \mathbb{R},$$

for some constant $C > 0$. Suppose that the Fourier sampling density ω satisfies (7.104). Suppose that $\mathbf{M} = (M_1, \ldots, M_r)$ corresponds to wavelet scales with $M_k = \mathcal{O}\left(2^{R_k}\right)$ for $k = 1, \ldots, r$ and $\mathbf{s} = (s_1, \ldots, s_r)$ corresponds to the sparsities within them. Let $\epsilon > 0$ and let $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ be a multilevel sampling scheme such that the following holds:

(i) For general Φ and Ψ, the parameters

$$N = N_r, \quad K = \max_{k=1,\ldots,r} \left\{ \frac{N_k - N_{k-1}}{m_k} \right\}, \quad M = M_r, \quad s = s_1 + \ldots + s_r$$

satisfy $N \gtrsim M^{1+1/(2\alpha-1)} \cdot (\log_2(4MK\sqrt{s}))^{1/(2\alpha-1)}$. If we additionally assume that

$$\left|\hat{\Phi}^{(k)}(\xi)\right| \leq \frac{C}{(1+|\xi|)^{\alpha}}, \quad \left|\hat{\Psi}^{(k)}(\xi)\right| \leq \frac{C}{(1+|\xi|)^{\alpha}}, \quad \xi \in \mathbb{R}, \quad k = 0, 1, 2, \quad \alpha \geq 1.5, \tag{6.1}$$

where $\hat{\Phi}^{(k)}$ and $\hat{\Psi}^{(k)}$ denotes the kth derivative of the Fourier transform of Φ and Ψ respectively, then it suffices to let $N \gtrsim M \cdot (\log_2(4KM\sqrt{s}))^{1/(2\alpha-1)}$.

(ii) For $k = 1, \ldots, r-1$, $N_k = 2^{R_k}\omega^{-1}$.

(iii) For each $k = 1, \ldots, r$,

$$m_k \gtrsim \log(\epsilon^{-1}) \cdot \log\left((K\sqrt{s})^{1+1/v} N\right) \cdot \frac{N_k - N_{k-1}}{N_{k-1}} \cdot \left(\hat{s}_k + \sum_{l=1}^{k-2} s_l \cdot 2^{-\alpha(R_{k-1}-R_l)} + \sum_{l=k+2}^{r} s_l \cdot 2^{-v(R_{l-1}-R_k)}\right) \tag{6.2}$$

where $\hat{s}_k = \max\{s_{k-1}, s_k, s_{k+1}\}$.

Then, with probability exceeding $1 - s\epsilon$, any minimizer $\xi \in l^1(\mathbb{N})$ of (4.1) satisfies

$$\|\xi - x\| \leq C \cdot \left(\delta \cdot \sqrt{K} \cdot \left(1 + L \cdot \sqrt{s}\right) + \sigma_{\mathbf{s},\mathbf{M}}(f)\right),$$

for some constant C, where $\sigma_{\mathbf{s},\mathbf{M}}(f)$ is as in (3.1), and $L = C \cdot \left(1 + \frac{\sqrt{\log_2(6\epsilon^{-1})}}{\log_2(4KM\sqrt{s})}\right)$. If $m_k = N_k - N_{k-1}$ for $1 \leq k \leq r$ then this holds with probability 1.

This theorem shows near-optimal recovery of wavelet coefficients from Fourier samples when using multilevel sampling. It therefore provides the first comprehensive explanation for the success of compressed sensing in the aforementioned applications. To see this, consider the key estimate (6.2). This shows that m_k need only scale as a linear combination of the local sparsities s_l, $1 \leq l \leq r$. Critically, the dependence of the sparsities s_l for $l \neq k$ is exponentially diminishing in $|k - l|$. Note that the presence of the off-diagonal terms is due to the previously-discussed phenomenon of interference, which occurs since the Fourier/wavelets system is not exactly block diagonal. Nonetheless, the system is nearly block-diagonal, and this results in the near-optimality seen in (6.2).

Remark 6.1 The Fourier/wavelets recovery problem was studied by Candès & Romberg in [15]. Their result shows that if an image can be first separated into separate wavelet subbands before sampling, then it can be recovered using approximately s_k measurements (up to a log factor) in each sampling band. Unfortunately, separation into separate wavelet subbands is infeasible in most practical situations. Theorem 6.2 improves on this result by removing this restriction, with the sole penalty being the slightly worse bound (6.2).

A recovery result for bivariate Haar wavelets, as well as the related technique of TV minimization, was given in [47]. Similarly [10] analyzes block sampling strategies with application to MRI. However, these results are based on sparsity, and therefore they do not explain how the sampling strategy will depend on the signal structure.

7 Proofs

The proofs rely on some key propositions from which one can deduce the main theorems. The main work is to prove these proposition, and that will be done subsequently.

7.1 Key results

Proposition 7.1. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ and suppose that Δ and $\Omega = \Omega_1 \cup \ldots \cup \Omega_r$ (where the union is disjoint) are subsets of $\mathbb{N}$. Let $x_0 \in \mathcal{H}$ and $z \in \mathrm{ran}(P_\Omega U)$ be such that $\|z\| \leq \delta$ for $\delta \geq 0$. Let $M \in \mathbb{N}$ and $y = P_\Omega U x_0 + z$ and $y_M = P_\Omega U P_M x_0 + z$. Suppose that $\xi \in \mathcal{H}$ and $\xi_M \in \mathcal{H}$ satisfiy*

$$\|\xi\|_{l^1} = \inf_{\eta \in \mathcal{H}} \{\|\eta\|_{l^1} : \|P_\Omega U \eta - y\| \leq \delta\}. \tag{7.1}$$

$$\|\xi_M\|_{l^1} = \inf_{\eta \in \mathbb{C}^M} \{\|\eta\|_{l^1} : \|P_\Omega U P_M \eta - y_M\| \leq \delta\}. \tag{7.2}$$

If there exists a vector $\rho = U^* P_\Omega w$ such that

(i) $\|P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r}\right) U P_\Delta - I_\Delta\| \leq \frac{1}{4}$

(ii) $\max_{i \in \Delta^c} \| \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} P_{\Omega_r}\right) U e_i\| \leq \sqrt{\frac{5}{4}}$

(iii) $\|P_\Delta \rho - \mathrm{sgn}(P_\Delta x_0)\| \leq \frac{q}{8}$.

(iv) $\|P_\Delta^\perp \rho\|_{l^\infty} \leq \frac{1}{2}$

(v) $\|w\| \leq L \cdot \sqrt{|\Delta|}$

for some $L > 0$ and $0 < q_k \leq 1$, $k = 1, \ldots, r$, then we have that

$$\|\xi - x_0\| \leq C \cdot \left(\delta \cdot \left(\frac{1}{\sqrt{q}} + L\sqrt{s} \right) + \|P_\Delta^\perp x_0\|_{l^1} \right),$$

for some constant C, where $s = |\Delta|$ and $q = \min\{q_k\}_{k=1}^r$. Also, if (ii) is replaced by

$$\max_{i \in \{1,\ldots,M\} \cap \Delta^c} \| \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} P_{\Omega_r} \right) U e_i \| \leq \sqrt{\frac{5}{4}}$$

and (iv) is replaced by $\|P_M P_\Delta^\perp \rho\|_{l^\infty} \leq \frac{1}{2}$ then

$$\|\xi_M - x_0\| \leq C \cdot \left(\delta \cdot \left(\frac{1}{\sqrt{q}} + L\sqrt{s} \right) + \|P_M P_\Delta^\perp x_0\|_{l^1} \right). \tag{7.3}$$

Proof. First observe that (i) implies that $(P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) U P_\Delta|_{P_\Delta(\mathcal{H})})^{-1}$ exists and

$$\|(P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) U P_\Delta|_{P_\Delta(\mathcal{H})})^{-1}\| \leq \frac{4}{3}. \tag{7.4}$$

Also, (i) implies that

$$\| \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} P_{\Omega_r} \right) U P_\Delta \|^2 = \|P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) U P_\Delta \| \leq \frac{5}{4}, \tag{7.5}$$

and

$$\begin{aligned}
&\|P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) \|^2 = \| \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) U P_\Delta \|^2 \\
&= \sup_{\|\eta\|=1} \| \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) U P_\Delta \eta \|^2 \\
&= \sup_{\|\eta\|=1} \sum_{k=1}^r \|q_k^{-1} P_{\Omega_k} U P_\Delta \eta\|^2 \leq \frac{1}{q} \sup_{\|\eta\|=1} \sum_{k=1}^r q_k^{-1} \|P_{\Omega_k} U P_\Delta \eta\|^2, \quad \frac{1}{q} = \max_{1 \leq k \leq r} \{\frac{1}{q_k}\} \\
&= \frac{1}{q} \sup_{\|\eta\|=1} \langle P_\Delta U^* \left(\sum_{k=1}^r q_k^{-1} P_{\Omega_k} \right) U P_\Delta \eta, \eta \rangle \leq \frac{1}{q} \|P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) U P_\Delta \|.
\end{aligned} \tag{7.6}$$

Thus, (7.5) and (7.6) imply

$$\|P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) \| \leq \sqrt{\frac{5}{4q}}. \tag{7.7}$$

Suppose that there exists a vector ρ, constructed with $y_0 = P_\Delta x_0$, satisfying (iii)-(v). Let ξ be a solution to (7.1) and let $h = \xi - x_0$. Let $A_\Delta = P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) U P_\Delta|_{P_\Delta(\mathcal{H})}$. Then, it follows from (ii) and observations (7.4), (7.5), (7.7) that

$$\begin{aligned}
&\|P_\Delta h\| = \|A_\Delta^{-1} A_\Delta P_\Delta h\| \\
&\leq \|A_\Delta^{-1}\| \|P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) U (I - P_\Delta^\perp) h\| \\
&\leq \frac{4}{3} \|P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) \| \|P_\Omega U h\| \\
&+ \frac{4}{3} \max_{i \in \Delta^c} \|P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) U e_i\| \|P_\Delta^\perp h\|_{l^1} \\
&\leq \frac{4}{3} \|P_\Delta U^* \left(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r} \right) \| \|P_\Omega U h\| \\
&+ \frac{4}{3} \left\| P_\Delta U^* \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} \right) \right\| \max_{i \in \Delta^c} \left\| \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} P_{\Omega_r} \right) U e_i \right\| \|P_\Delta^\perp h\|_{l^1} \\
&\leq \frac{4\sqrt{5}}{3\sqrt{q}} \delta + \frac{5}{3} \|P_\Delta^\perp h\|_{l^1},
\end{aligned} \tag{7.8}$$

where in the final step we use $\|P_\Omega U h\| \leq \|P_\Omega U\zeta - y\| + \|z\| \leq 2\delta$. We will now obtain a bound for $\|P_\Delta^\perp h\|_{l^1}$. First note that

$$\begin{aligned}\|h + x_0\|_{l^1} &= \|P_\Delta h + P_\Delta x_0\|_{l^1} + \|P_\Delta^\perp (h + x_0)\|_{l^1} \\ &\geq \operatorname{Re}\langle P_\Delta h, \operatorname{sgn}(P_\Delta x_0)\rangle + \|P_\Delta x_0\|_{l^1} + \|P_\Delta^\perp h\|_{l^1} - \|P_\Delta^\perp x_0\|_{l^1} \\ &\geq \operatorname{Re}\langle P_\Delta h, \operatorname{sgn}(P_\Delta x_0)\rangle + \|x_0\|_{l^1} + \|P_\Delta^\perp h\|_{l^1} - 2\|P_\Delta^\perp x_0\|_{l^1}.\end{aligned} \tag{7.9}$$

Since $\|x_0\|_{l^1} \geq \|h + x_0\|_{l^1}$, we have that

$$\|P_\Delta^\perp h\|_{l^1} \leq |\langle P_\Delta h, \operatorname{sgn}(P_\Delta x_0)\rangle| + 2\|P_\Delta^\perp x_0\|_{l^1}. \tag{7.10}$$

We will use this equation later on in the proof, but before we do that observe that some basic adding and subtracting yields

$$\begin{aligned}|\langle P_\Delta h, \operatorname{sgn}(x_0)\rangle| &\leq |\langle P_\Delta h, \operatorname{sgn}(P_\Delta x_0) - P_\Delta \rho\rangle| + |\langle h, \rho\rangle| + \left|\langle P_\Delta^\perp h, P_\Delta^\perp \rho\rangle\right| \\ &\leq \|P_\Delta h\|\|\operatorname{sgn}(P_\Delta x_0) - P_\Delta \rho\| + |\langle P_\Omega U h, w\rangle| + \|P_\Delta^\perp h\|_{l^1}\|P_\Delta^\perp \rho\|_{l^\infty} \\ &\leq \frac{q}{8}\|P_\Delta h\| + 2L\delta\sqrt{s} + \frac{1}{2}\|P_\Delta^\perp h\|_{l^1} \\ &\leq \frac{\sqrt{5q}}{6}\delta + \frac{5q}{24}\|P_\Delta^\perp h\|_{l^1} + 2L\delta\sqrt{s} + \frac{1}{2}\|P_\Delta^\perp h\|_{l^1}\end{aligned} \tag{7.11}$$

where the last inequality utilises (7.8) and the penultimate inequality follows from properties (iii), (iv) and (v) of the dual vector ρ. Combining this with (7.10) and the fact that $q \leq 1$ gives that

$$\|P_\Delta^\perp h\|_{l^1} \leq \delta\left(\frac{4\sqrt{5q}}{3} + 8L\sqrt{s}\right) + 8\|P_\Delta^\perp x_0\|_{l^1}. \tag{7.12}$$

Thus, (7.8) and (7.12) yields:

$$\|h\| \leq \|P_\Delta h\| + \left\|P_\Delta^\perp h\right\| \leq \frac{8}{3}\|P_\Delta^\perp h\|_{l^1} + \frac{4\sqrt{5}}{3\sqrt{q}}\delta \leq \left(8\sqrt{q} + 22L\sqrt{s} + \frac{3}{\sqrt{q}}\right)\cdot\delta + 22\left\|P_\Delta^\perp x_0\right\|_{l^1}. \tag{7.13}$$

The proof of the second part of this proposition follows the proof as outlined above and we omit the details. □

The next two propositions give sufficient conditions for Proposition 7.1 to be true. But before we state them we need to define the following.

Definition 7.2. *Let U be an isometry of either $\mathbb{C}^{N\times N}$ or $\mathcal{B}(l^2(\mathbb{N}))$. For $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$, $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$ with $1 \leq N_1 < \ldots < N_r$ and $1 \leq M_1 < \ldots < M_r$, $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $1 \leq k \leq r$, let*

$$\kappa_{\mathbf{N},\mathbf{M}}(k, l) = \max_{\eta\in\Theta} \|P_{N_k}^{N_{k-1}} U P_{M_l}^{M_{l-1}} \eta\|_{l^\infty} \cdot \sqrt{\mu(P_{N_k}^{N_{k-1}} U)}.$$

where

$$\Theta = \{\eta : \|\eta\|_{l^\infty} \leq 1, |\operatorname{supp}(P_{M_l}^{M_{l-1}}\eta)| = s_l,\ l = 1, \ldots, r-1,\ |\operatorname{supp}(P_{M_{r-1}}^\perp \eta)| = s_r, \},$$

and $N_0 = M_0 = 0$. We also define

$$\kappa_{\mathbf{N},\mathbf{M}}(k, \infty) = \max_{\eta\in\Theta} \|P_{N_k}^{N_{k-1}} U P_{M_{r-1}}^\perp \eta\|_{l^\infty} \cdot \sqrt{\mu(P_{N_k}^{N_{k-1}} U)}.$$

Proposition 7.3. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ be an isometry and $x \in l^1(\mathbb{N})$. Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a multilevel sampling scheme, where $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Let $(\mathbf{s}, \mathbf{M})$, where $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$, $M_1 < \ldots < M_r$, and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, be any pair such that the following holds:*

(i) The parameters $N := N_r$, and $K := \max_{k=1,\ldots,r}(N_k - N_{k-1})/m_k$, satisfy the weak balancing property with respect to U, $M := M_r$ and $s := s_1 + \ldots + s_r$;

(ii) for $\epsilon > 0$ and $1 \leq k \leq r$,

$$1 \gtrsim (\log(s\epsilon^{-1}) + 1) \cdot \frac{N_k - N_{k-1}}{m_k} \cdot \left(\sum_{l=1}^{r} \kappa_{\mathbf{N},\mathbf{M}}(k,l) \right) \cdot \log\left(KM\sqrt{s}\right), \tag{7.14}$$

(iii)

$$m_k \gtrsim (\log(s\epsilon^{-1}) + 1) \cdot \hat{m}_k \cdot \log\left(KM\sqrt{s}\right), \tag{7.15}$$

where $\hat{m}_k$ satisfies

$$1 \gtrsim \sum_{k=1}^{r} \left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1 \right) \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot \tilde{s}_k, \qquad \forall\, l = 1, \ldots, r,$$

where $\tilde{s}_1 + \ldots + \tilde{s}_r \leq s_1 + \ldots + s_r$, $\tilde{s}_k \leq S_k(s_1, \ldots, s_r)$ and S_k is defined in (4.3).

Then (i)-(v) in Proposition 7.1 follow with probability exceeding $1 - \epsilon$, with (ii) replaced by

$$\max_{i \in \{1,\ldots,M\} \cap \Delta^c} \left\| \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} P_{\Omega_r} \right) U e_i \right\| \leq \sqrt{\frac{5}{4}}, \tag{7.16}$$

(iv) replaced by $\|P_M P_\Delta^\perp \rho\|_{l^\infty} \leq \frac{1}{2}$ and L in (v) is given by

$$L = C \cdot \sqrt{K} \cdot \left(1 + \frac{\sqrt{\log_2(6\epsilon^{-1})}}{\log_2(4KM\sqrt{s})} \right). \tag{7.17}$$

If $m_k = N_k - N_{k-1}$ for all $1 \leq k \leq r$ then (i)-(v) follow with probability one (with the alterations suggested above).

Proposition 7.4. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ be an isometry and $x \in l^1(\mathbb{N})$. Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a multilevel sampling scheme, where $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Let $(\mathbf{s}, \mathbf{M})$, where $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$, $M_1 < \ldots < M_r$, and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, be any pair such that the following holds:*

(i) The parameters N and K (as in Proposition 7.3) satisfy the strong balancing property with respect to U, $M = M_r$ and $s := s_1 + \ldots + s_r$;

(ii) for $\epsilon > 0$ and $1 \leq k \leq r$,

$$1 \gtrsim (\log(s\epsilon^{-1}) + 1) \cdot \frac{N_k - N_{k-1}}{m_k} \cdot \left(\kappa_{\mathbf{N},\mathbf{M}}(k,\infty) + \sum_{l=1}^{r-1} \kappa_{\mathbf{N},\mathbf{M}}(k,l) \right) \cdot \log\left(K\tilde{M}\sqrt{s}\right), \tag{7.18}$$

(iii)

$$m_k \gtrsim (\log(s\epsilon^{-1}) + 1) \cdot \hat{m}_k \cdot \log\left(K\tilde{M}\sqrt{s}\right), \tag{7.19}$$

where $\tilde{M} = \min\{i \in \mathbb{N} : \| \max_{j \geq i} P_N U P_{\{j\}} \| \leq 1/(K32\sqrt{s})\}$, and $\hat{m}_k$ is as in Proposition 7.3.

Then (i)-(v) in Proposition 7.1 follow with probability exceeding $1-\epsilon$ with L as in (7.17). If $m_k = N_k - N_{k-1}$ for all $1 \leq k \leq r$ then (i)-(v) follow with probability one.

Lemma 7.5 (Bounds for $\kappa_{\mathbf{N},\mathbf{M}}(k,l)$)**.** *For $k, l = 1, \ldots, r$*

$$\kappa_{\mathbf{N},\mathbf{M}}(k,l) \leq \min\left\{ \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot s_l,\ \sqrt{s_l \cdot \mu(P_{N_k}^{N_{k-1}} U)} \cdot \left\| P_{N_k}^{N_{k-1}} U P_{M_l}^{M_{l-1}} \right\| \right\}. \tag{7.20}$$

Also, for $k = 1, \ldots, r$

$$\kappa_{\mathbf{N},\mathbf{M}}(k,\infty) \leq \min\left\{ \mu_{\mathbf{N},\mathbf{M}}(k,\infty) \cdot s_r,\ \sqrt{s_r \cdot \mu(P_{N_k}^{N_{k-1}} U)} \cdot \left\| P_{N_k}^{N_{k-1}} U P_{M_{r-1}}^{\perp} \right\| \right\}. \tag{7.21}$$

Proof. For $k, l = 1, \ldots, r$

$$\begin{aligned}\kappa_{\mathbf{N},\mathbf{M}}(k,l) &= \max_{\eta\in\Theta} \|P_{N_k}^{N_{k-1}} U P_{M_l}^{M_{l-1}} \eta\|_{l^\infty} \cdot \sqrt{\mu(P_{N_k}^{N_{k-1}} U)} \\ &= \max_{\eta\in\Theta} \max_{N_{k-1}<i\leq N_k} \left| \sum_{M_{l-1}<j\leq M_l} \eta_j u_{ij} \right| \cdot \sqrt{\mu(P_{N_k}^{N_{k-1}} U)} \\ &\leq s_l \cdot \sqrt{\mu(P_{N_k}^{N_{k-1}} U P_{M_l}^{M_{l-1}})} \cdot \sqrt{\mu(P_{N_k}^{N_{k-1}} U)} \leq s_l \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l)\end{aligned}$$

since $|u_{ij}| \leq 1$, and similarly,

$$\begin{aligned}\kappa_{\mathbf{N},\mathbf{M}}(k,\infty) &= \max_{\eta\in\Theta} \|P_{N_k}^{N_{k-1}} U P_{M_{r-1}}^{\perp} \eta\|_{l^\infty} \cdot \sqrt{\mu(P_{N_k}^{N_{k-1}} U)} \\ &= \max_{\eta\in\Theta} \max_{N_{k-1}<i\leq N_k} \left| \sum_{M_{r-1}<j} \eta_j u_{ij} \right| \cdot \sqrt{\mu(P_{N_k}^{N_{k-1}} U)} \leq s_r \cdot \mu_{\mathbf{N},\mathbf{M}}(k,\infty).\end{aligned}$$

Finally, it is straightforward to show that for $k, l = 1, \ldots, r$,

$$\kappa_{\mathbf{N},\mathbf{M}}(k,l) \leq \sqrt{s_l} \cdot \left\| P_{N_k}^{N_{k-1}} U P_{M_l}^{M_{l-1}} \right\| \sqrt{\mu(P_{N_k}^{N_{k-1}} U)}$$

and

$$\kappa_{\mathbf{N},\mathbf{M}}(k,\infty) \leq \sqrt{s_r} \cdot \left\| P_{N_k}^{N_{k-1}} U P_{M_{r-1}}^{\perp} \right\| \sqrt{\mu(P_{N_k}^{N_{k-1}} U)}.$$

□

We are now ready to prove the main theorems.

Proof of Theorems 4.1 and 5.2. It is clear that Theorem 4.1 follows from Theorem 5.2, thus it remains to prove the latter. We will apply Proposition 7.3 to a two-level sampling scheme $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$, where $\mathbf{N} = (N_1, N_2)$ and $\mathbf{m} = (m_1, m_2)$ with $m_1 = N_1$ and $m_2 = m$. Also, consider $(\mathbf{s}, \mathbf{M})$, where $\mathbf{s} = (M_1, s_2)$, $\mathbf{M} = (M_1, M_2)$. Thus, if $N_1, N_2, m_1, m_2 \in \mathbb{N}$ are such that

$$N = N_2, \quad K = \max\left\{ \frac{N_2 - N_1}{m_2}, \frac{N_1}{m_1} \right\}$$

satisfy the weak balancing property with respect to U, $M = M_2$ and $s = M_1 + s_2$, we have that (i) - (v) in Proposition 7.1 follow with probability exceeding $1 - s\epsilon$, with (ii) replaced by

$$\max_{i\in\{1,\ldots,M\}\cap\Delta^c} \| \left(P_{N_1} \oplus \frac{N_2 - N_1}{m_2} P_{\Omega_2} \right) U e_i \| \leq \sqrt{\frac{5}{4}},$$

(iv) replaced by $\|P_M P_\Delta^\perp \rho\|_{l^\infty} \leq \frac{1}{2}$ and L in (v) is given by (7.17), if

$$1 \gtrsim (\log(s\epsilon^{-1}) + 1) \cdot \frac{N - N_1}{m_2} \cdot (\kappa_{\mathbf{N},\mathbf{M}}(2,1) + \kappa_{\mathbf{N},\mathbf{M}}(2,2)) \cdot \log\left(KM\sqrt{s}\right), \tag{7.22}$$

$$m_2 \gtrsim (\log(s\epsilon^{-1}) + 1) \cdot \hat{m}_2 \cdot \log\left(KM\sqrt{s}\right), \tag{7.23}$$

where $\hat{m}_2$ satisfies $1 \gtrsim ((N_2 - N_1)/\hat{m}_2 - 1) \cdot \mu_{N_1} \cdot \tilde{s}_2$, and $\tilde{s}_2 \leq S_2$ (recall S_2 from Definition 4.3). Recall from (7.20) that

$$\kappa_{\mathbf{N},\mathbf{M}}(2,1) \leq \sqrt{s_1 \cdot \mu_{N_1}} \cdot \left\| P_{N_1}^\perp U P_{M_1} \right\|, \quad \kappa_{\mathbf{N},\mathbf{M}}(2,2) \leq s_2 \cdot \mu_{N_1}.$$

Also, it follows directly from Definition 4.3 that

$$S_2 \leq \left(\left\| P_{N_1}^\perp U P_{M_1} \right\| \cdot \sqrt{M_1} + \sqrt{s_2} \right)^2.$$

Thus, provided that $\left\| P_{N_1}^\perp U P_{M_1} \right\| \leq \gamma/\sqrt{M_1}$ where γ is as in (i) of Theorem 5.2, we observe that (iii) of Theorem 5.2 implies (7.22) and (7.23). Thus, the theorem now follows from Proposition 7.1. □

Proof of Theorem 4.4 and Theorem 5.3. It is straightforward that Theorem 4.4 follows from Theorem 5.3. Now, recall from Lemma 7.20 that

$$\kappa_{\mathbf{N},\mathbf{M}}(k,l) \leq s_l \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l), \quad \kappa_{\mathbf{N},\mathbf{M}}(k,\infty) \leq s_r \cdot \mu_{\mathbf{N},\mathbf{M}}(k,\infty), \qquad k,l = 1,\ldots,r.$$

Thus, a direct application of Proposition 7.4 and Proposition 7.1 completes the proof.

□

It remains now to prove Propositions 7.3 and 7.4. This is the content of the next sections.

7.2 Preliminaries

Before we commence on the rather length proof of these propositions, let us recall one of the monumental results in probability theory that will be of greater use later on.

Theorem 7.6. *(Talagrand [69, 52]) There exists a number K with the following property. Consider n independent random variables X_i valued in a measurable space Ω and let $\mathcal{F}$ be a (countable) class of measurable functions on Ω. Let Z be the random variable $Z = \sup_{f\in\mathcal{F}} \sum_{i\leq n} f(X_i)$ and define*

$$S = \sup_{f\in\mathcal{F}} \|f\|_\infty, \qquad V = \sup_{f\in\mathcal{F}} \mathbb{E}\left(\sum_{i\leq n} f(X_i)^2\right).$$

If $\mathbb{E}(f(X_i)) = 0$ for all $f \in \mathcal{F}$ and $i \leq n$, then, for each $t > 0$, we have

$$\mathbb{P}(|Z - \mathbb{E}(Z)| \geq t) \leq 3\exp\left(-\frac{1}{K}\frac{t}{S}\log\left(1 + \frac{tS}{V + S\mathbb{E}(\overline{Z})}\right)\right),$$

where $\overline{Z} = \sup_{f\in\mathcal{F}} |\sum_{i\leq n} f(X_i)|$.

Note that this version of Talagrand's theorem is found in [52, Cor. 7.8]. We next present a theorem and several technical propositions that will serve as the main tools in our proofs of Propositions 7.3 and 7.4. A crucial tool herein is the Bernoulli sampling model. We will use the notation $\{a,\ldots,b\} \supset \Omega \sim \mathrm{Ber}(q)$, where $a < b$ $a,b \in \mathbb{N}$, when Ω is given by $\Omega = \{k : \delta_k = 1\}$ and $\{\delta_k\}_{k=1}^N$ is a sequence of Bernoulli variables with $\mathbb{P}(\delta_k = 1) = q$.

Definition 7.7. *Let $r \in \mathbb{N}$, $\mathbf{N} = (N_1,\ldots,N_r) \in \mathbb{N}^r$ with $1 \leq N_1 < \ldots < N_r$, $\mathbf{m} = (m_1,\ldots,m_r) \in \mathbb{N}^r$, with $m_k \leq N_k - N_{k-1}$, $k = 1,\ldots,r$, and suppose that*

$$\Omega_k \subseteq \{N_{k-1}+1,\ldots,N_k\}, \quad \Omega_k \sim \mathrm{Ber}\left(\frac{m_k}{N_k - N_{k-1}}\right), \quad k = 1,\ldots,r,$$

where $N_0 = 0$. We refer to the set

$$\Omega = \Omega_{\mathbf{N},\mathbf{m}} := \Omega_1 \cup \ldots \cup \Omega_r.$$

as an $(\mathbf{N},\mathbf{m})$-multilevel Bernoulli sampling scheme.

Theorem 7.8. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ be an isometry. Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a multilevel Bernoulli sampling scheme, where $\mathbf{N} = (N_1,\ldots,N_r) \in \mathbb{N}^r$ and $\mathbf{m} = (m_1,\ldots,m_r) \in \mathbb{N}^r$. Consider $(\mathbf{s},\mathbf{M})$, where $\mathbf{M} = (M_1,\ldots,M_r) \in \mathbb{N}^r$, $M_1 < \ldots < M_r$, and $\mathbf{s} = (s_1,\ldots,s_r) \in \mathbb{N}^r$, and let*

$$\Delta = \Delta_1 \cup \ldots \cup \Delta_r, \qquad \Delta_k \subset \{M_{k-1}+1,\ldots,M_k\}, \qquad |\Delta_k| = s_k$$

where $M_0 = 0$. If $\|P_{M_r}U^*P_{N_r}UP_{M_r} - P_{M_r}\| \leq 1/8$ then, for $\gamma \in (0,1)$,

$$\mathbb{P}(\|P_\Delta U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r})UP_\Delta - P_\Delta\| \geq 1/4) \leq \gamma, \tag{7.24}$$

where $q_k = m_k/(N_k - N_{k-1})$, provided that

$$1 \gtrsim \frac{N_k - N_{k-1}}{m_k} \cdot \left(\sum_{l=1}^r \kappa_{\mathbf{N},\mathbf{M}}(k,l)\right) \cdot \left(\log\left(\gamma^{-1}s\right) + 1\right). \tag{7.25}$$

In addition, if $q = \min\{q_k\}_{k=1}^r = 1$ then

$$\mathbb{P}(\|P_\Delta U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r})UP_\Delta - P_\Delta\| \geq 1/4) = 0.$$

In proving this theorem we deliberately avoid the use of the Matrix Bernstein inequality [39], as Talagrand's theorem is more convenient for our setting. Before we can prove this theorem, we need the following technical lemma.

Lemma 7.9. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ with $\|U\| \leq 1$, and consider the setup in Theorem 7.8. Let $N = N_r$ and let $\{\delta_j\}_{j=1}^N$ be independent random Bernoulli variables with $\mathbb{P}(\delta_j = 1) = \tilde{q}_j$, $\tilde{q}_j = m_k/(N_k - N_{k-1})$ and $j \in \{N_{k-1}+1, \ldots, N_k\}$, and define $Z = \sum_{j=1}^N Z_j$, $Z_j = \left(\tilde{q}_j^{-1}\delta_j - 1\right)\eta_j \otimes \bar{\eta}_j$ and $\eta_j = P_\Delta U^* e_j$. Then*

$$\mathbb{E}\left(\|Z\|\right)^2 \leq 48 \max\{\log(|\Delta|), 1\} \max_{1\leq j\leq N} \left\{\tilde{q}_j^{-1}\|\eta_j\|^2\right\},$$

when $\left(\max\{\log(|\Delta|), 1\}\right)^{-1} \geq 18 \max_{1\leq j\leq N}\left\{\tilde{q}_j^{-1}\|\eta_j\|^2\right\}$.

The proof of this lemma involves essentially reworking an argument due to Rudelson [64], and is similar to arguments given previously in [1] (see also [15]). We include it here for completeness as the setup deviates slightly. We shall also require the following result:

Lemma 7.10. *(Rudelson) Let $\eta_1, \ldots, \eta_M \in \mathbb{C}^n$ and let $\varepsilon_1, \ldots \varepsilon_M$ be independent Bernoulli variables taking values $1, -1$ with probability $1/2$. Then*

$$\mathbb{E}\left(\left\|\sum_{i=1}^M \varepsilon_i \bar{\eta}_i \otimes \eta_i\right\|\right) \leq \frac{3}{2}\sqrt{p}\max_{i\leq M}\|\eta_i\|\sqrt{\left\|\sum_{i=1}^M \bar{\eta}_i \otimes \eta_i\right\|},$$

where $p = \max\{2, 2\log(n)\}$.

Lemma 7.10 is often referred to as Rudelson's Lemma [64]. However, we use the above complex version that was proven by Tropp [71, Lem. 22].

Proof of Lemma 7.9. We commence by letting $\tilde{\delta} = \{\tilde{\delta}_j\}_{j=1}^N$ be independent copies of $\delta = \{\delta_j\}_{j=1}^N$. Then, since $\mathbb{E}(Z) = 0$,

$$\begin{aligned}\mathbb{E}_\delta\left(\|Z\|\right) &= \mathbb{E}_\delta\left(\left\|Z - \mathbb{E}_{\tilde{\delta}}\left(\sum_{j=1}^N \left(\tilde{q}_j^{-1}\tilde{\delta}_j - 1\right)\eta_j \otimes \bar{\eta}_j\right)\right\|\right) \\ &\leq \mathbb{E}_\delta\left(\mathbb{E}_{\tilde{\delta}}\left(\left\|Z - \sum_{j=1}^N \left(\tilde{q}_j^{-1}\tilde{\delta}_j - 1\right)\eta_j \otimes \bar{\eta}_j\right\|\right)\right),\end{aligned} \tag{7.26}$$

by Jensen's inequality. Let $\varepsilon = \{\varepsilon_j\}_{j=1}^N$ be a sequence of Bernoulli variables taking values ± 1 with probability $1/2$. Then, by (7.26), symmetry, Fubini's Theorem and the triangle inequality, it follows that

$$\begin{aligned}\mathbb{E}_\delta\left(\|Z\|\right) &\leq \mathbb{E}_\varepsilon\left(\mathbb{E}_\delta\left(\mathbb{E}_{\tilde{\delta}}\left(\left\|\sum_{j=1}^N \varepsilon_j\left(\tilde{q}_j^{-1}\delta_j - \tilde{q}_j^{-1}\tilde{\delta}_j\right)\eta_j \otimes \bar{\eta}_j\right\|\right)\right)\right) \\ &\leq 2\mathbb{E}_\delta\left(\mathbb{E}_\varepsilon\left(\left\|\sum_{j=1}^N \varepsilon_j \tilde{q}_j^{-1}\delta_j \eta_j \otimes \bar{\eta}_j\right\|\right)\right).\end{aligned} \tag{7.27}$$

We are now able to apply Rudelson's Lemma (Lemma 7.10). However, as specified before, it is the complex version that is crucial here. By Lemma 7.10 we get that

$$\mathbb{E}_\varepsilon\left(\left\|\sum_{j=1}^N \varepsilon_j \tilde{q}_j^{-1}\delta_j \eta_j \otimes \bar{\eta}_j\right\|\right) \leq \frac{3}{2}\sqrt{\max\{2\log(s), 2\}}\max_{1\leq j\leq N}\tilde{q}_j^{-1/2}\|\eta_j\|\sqrt{\left\|\sum_{j=1}^N q_j^{-1}\tilde{q}_j^{-1}\delta_j\eta_j \otimes \bar{\eta}_j\right\|}, \tag{7.28}$$

where $s = |\Delta|$. And hence, by using (7.27) and (7.28), it follows that

$$\mathbb{E}_\delta\left(\|Z\|\right) \leq 3\sqrt{\max\{2\log(s), 2\}}\max_{1\leq j\leq N}\tilde{q}_j^{-1/2}\|\eta_j\|\sqrt{\mathbb{E}_\delta\left(\left\|Z + \sum_{j=1}^N \eta_j \otimes \bar{\eta}_j\right\|\right)}.$$

Note that $\|\sum_{j=1}^N \eta_j \otimes \bar{\eta}_j\| \leq 1$, since U is an isometry. The result now follows from the straightforward calculus fact that if $r > 0$, $c \leq 1$ and $r \leq c\sqrt{r+1}$ then we have that $r \leq c(1+\sqrt{5})/2$. $\square$

Proof of Theorem 7.8. Let $N = N_r$ just to be clear here. Let $\{\delta_j\}_{j=1}^N$ be random Bernoulli variables as defined in Lemma 7.9 and define $Z = \sum_{j=1}^N Z_j$, $Z_j = \left(\tilde{q}_j^{-1}\delta_j - 1\right)\eta_j \otimes \bar{\eta}_j$ with $\eta_j = P_\Delta U^* e_j$. Now observe that

$$P_\Delta U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r})UP_\Delta = \sum_{j=1}^N \tilde{q}_j^{-1}\delta_j \eta_j \otimes \bar{\eta}_j, \quad P_\Delta U^* P_N U P_\Delta = \sum_{j=1}^N \eta_j \otimes \bar{\eta}_j. \tag{7.29}$$

Thus, it follows that

$$\begin{aligned} \|P_\Delta U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r})UP_\Delta - P_\Delta\| &\leq \|Z\| + \|(P_\Delta U^* P_N U P_\Delta - P_\Delta)\| \\ &\leq \|Z\| + \frac{1}{8}, \end{aligned} \tag{7.30}$$

by the assumption that $\|P_{M_r}U^*P_{N_r}UP_{M_r} - P_{M_r}\| \leq 1/8$. Thus, to prove the assertion we need to estimate $\|Z\|$, and Talagrand's Theorem (Theorem 7.6) will be our main tool. Note that clearly, since Z is self-adjoint, we have that $\|Z\| = \sup_{\zeta\in\mathcal{G}} |\langle Z\zeta, \zeta\rangle|$, where $\mathcal{G}$ is a countable set of vectors in the unit ball of $P_\Delta(\mathcal{H})$. For $\zeta \in \mathcal{G}$ define the mappings

$$\hat{\zeta}_1(T) = \langle T\zeta, \zeta\rangle, \quad \hat{\zeta}_2(T) = -\langle T\zeta, \zeta\rangle, \qquad T \in \mathcal{B}(\mathcal{H}).$$

In order to use Talagrand's Theorem 7.6 we restrict the domain $\mathcal{D}$ of the mappings ζ_i to

$$\mathcal{D} = \{T \in \mathcal{B}(\mathcal{H}) : \|T\| \leq \max_{1\leq j\leq N}\{\tilde{q}_j^{-1}\|\eta_j\|^2\}\}.$$

Let $\mathcal{F}$ denote the family of mappings $\hat{\zeta}_1, \hat{\zeta}_2$ for $\zeta \in \mathcal{G}$. Then $\|Z\| = \sup_{\hat{\zeta}\in\mathcal{F}} \hat{\zeta}(Z)$, and for $i = 1, 2$ we have

$$|\hat{\zeta}_i(Z_j)| = \left|\left(\tilde{q}_j^{-1}\delta_j - 1\right)\right| |\langle(\eta_j \otimes \bar{\eta}_j)\,\zeta, \zeta\rangle| \leq \max_{1\leq j\leq N}\{\tilde{q}_j^{-1}\|\eta_j\|^2\}.$$

Thus, $Z_j \in \mathcal{D}$ for $1 \leq j \leq N$ and $S := \sup_{\zeta\in\mathcal{F}} \|\hat{\zeta}\|_\infty = \max_{1\leq j\leq N}\{\tilde{q}_j^{-1}\|\eta_j\|^2\}$. Note that

$$\|\eta_j\|^2 = \langle P_\Delta U^* e_j, P_\Delta U^* e_j\rangle = \sum_{k=1}^r \langle P_{\Delta_k} U^* e_j, P_{\Delta_k} U^* e_j\rangle.$$

Also, note that an easy application of Holder's inequality gives the following (note that the l^1 and l^∞ bounds are finite because all the projections have finite rank),

$$\begin{aligned} &|\langle P_{\Delta_k} U^* e_j, P_{\Delta_k} U^* e_j\rangle| \leq \|P_{\Delta_k} U^* e_j\|_{l^1} \|P_{\Delta_k} U^* e_j\|_{l^\infty} \\ &\leq \|P_{\Delta_k} U^* P_{N_l}^{N_{l-1}}\|_{l^1\to l^1} \|P_{\Delta_k} U^* e_j\|_{l^\infty} \leq \|P_{N_l}^{N_{l-1}} U P_{\Delta_k}\|_{l^\infty\to l^\infty} \cdot \sqrt{\mu(P_{N_l}^{N_{l-1}} U)} \leq \kappa_{\mathbf{N},\mathbf{M}}(l,k), \end{aligned}$$

for $j \in \{N_{l-1}+1, \ldots, N_l\}$ and $l \in \{1, \ldots, r\}$. Hence, it follows that

$$\|\eta_j\|^2 \leq \max_{1\leq k\leq r}(\kappa_{\mathbf{N},\mathbf{M}}(k,1) + \ldots + \kappa_{\mathbf{N},\mathbf{M}}(k,r)), \tag{7.31}$$

and therefore $S \leq \max_{1\leq k\leq r}\left(q_k^{-1}\sum_{j=1}^r \kappa_{\mathbf{N},\mathbf{M}}(k,j)\right)$. Finally, note that by (7.31) and the reasoning above, it follows that

$$\begin{aligned} V &:= \sup_{\hat{\zeta}_i\in\mathcal{F}} \mathbb{E}\left(\sum_{j=1}^N \hat{\zeta}_i(Z_j)^2\right) = \sup_{\zeta\in\mathcal{G}} \mathbb{E}\left(\sum_{j=1}^N \left(\tilde{q}_j^{-1}\delta_j - 1\right)^2 |\langle P_\Delta U^* e_j, \zeta\rangle|^4\right) \\ &\leq \max_{1\leq k\leq r} \|\eta_k\|^2 \left(\frac{N_k - N_{k-1}}{m_k} - 1\right) \sup_{\zeta\in\mathcal{G}} \sum_{j=1}^N |\langle e_j, UP_\Delta\zeta\rangle|^2, \\ &\leq \max_{1\leq k\leq r} \frac{N_k - N_{k-1}}{m_k}\left(\sum_{l=1}^r \kappa_{\mathbf{N},\mathbf{M}}(k,l)\right) \sup_{\zeta\in\mathcal{G}} \|U\zeta\|^2 = \max_{1\leq k\leq r} \frac{N_k - N_{k-1}}{m_k}\left(\sum_{l=1}^r \kappa_{\mathbf{N},\mathbf{M}}(k,l)\right), \end{aligned} \tag{7.32}$$

where we used the fact that U is an isometry to deduce that $\|U\| = 1$. Also, by Lemma 7.9 and (7.31) , it follows that

$$\mathbb{E}\left(\|Z\|\right)^2 \leq 48 \max_{1\leq k\leq r} \frac{N_k - N_{k-1}}{m_k} \left(\sum_{l=1}^{r} \kappa_{\mathbf{N},\mathbf{M}}(k,l)\right) \cdot \log(s) \tag{7.33}$$

when

$$1 \geq 18 \max_{1\leq k\leq r} \frac{N_k - N_{k-1}}{m_k} \left(\sum_{l=1}^{r} \kappa_{\mathbf{N},\mathbf{M}}(k,l)\right) \cdot \log(s), \tag{7.34}$$

(recall that we have assumed $s \geq 3$). Thus, by (7.30) and Talagrand's Theorem 7.6, it follows that

$$\begin{aligned}
&\mathbb{P}\left(\|P_\Delta U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r})UP_\Delta - P_\Delta\| \geq 1/4\right) \\
&\leq \mathbb{P}\left(\|Z\| \geq \frac{1}{16} + \sqrt{24 \max_{1\leq k\leq r} \frac{N_k - N_{k-1}}{m_k} \left(\sum_{l=1}^{r} \kappa_{\mathbf{N},\mathbf{M}}(k,l)\right) \cdot \log(s)}\right) \\
&\leq 3\exp\left(-\frac{1}{16K}\left(\max_{1\leq k\leq r} \frac{N_k - N_{k-1}}{m_k} \left(\sum_{l=1}^{r} \kappa_{\mathbf{N},\mathbf{M}}(k,l)\right)\right)^{-1} \log\left(1 + 1/32\right)\right),
\end{aligned} \tag{7.35}$$

when m_k's are chosen such that the right hand side of (7.33) is less than or equal to 1. Thus, by (7.30) and Talagrand's Theorem 7.6, it follows that

$$\begin{aligned}
&\mathbb{P}\left(\|P_\Delta U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r})UP_\Delta - P_\Delta\| \geq 1/4\right) \\
&\leq \mathbb{P}\left(\|Z\| \geq 1/8\right) \leq \mathbb{P}\left(\|Z\| \geq \frac{1}{16} + \mathbb{E}\|Z\|\right) \leq \mathbb{P}\left(|\|Z\| - \mathbb{E}\|Z\|| \geq \frac{1}{16}\right) \\
&\leq 3\exp\left(-\frac{1}{16K}\left(\max_{1\leq k\leq r} \frac{N_k - N_{k-1}}{m_k} \left(\sum_{l=1}^{r} \kappa_{\mathbf{N},\mathbf{M}}(k,l)\right)\right)^{-1} \log\left(1 + 1/32\right)\right),
\end{aligned} \tag{7.36}$$

when m_k's are chosen such that the right hand side of (7.33) is less than or equal to $1/16^2$. Note that this condition is implied by the assumptions of the theorem as is (7.34). This yields the first part of the theorem. The second claim of this theorem follows from the assumption that $\|P_{M_r}U^*P_{N_r}UP_{M_r} - P_{M_r}\| \leq 1/8$. □

Proposition 7.11. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ be an isometry. Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a multilevel Bernoulli sampling scheme, where $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Consider $(\mathbf{s}, \mathbf{M})$, where $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$, $M_1 < \ldots < M_r$, and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, and let*

$$\Delta = \Delta_1 \cup \ldots \cup \Delta_r, \qquad \Delta_k \subset \{M_{k-1}, \ldots, M_k\}, \qquad |\Delta_k| = s_k$$

where $M_0 = 0$. Let $\beta \geq 1/4$.

(i) If

$$N := N_r, \quad K := \max_{k=1,\ldots,r} \left\{\frac{N_k - N_{k-1}}{m_k}\right\},$$

satisfy the weak balancing property with respect to U, $M := M_r$ and $s := s_1 + \ldots + s_r$, then, for $\xi \in \mathcal{H}$ and $\beta, \gamma > 0$, we have that

$$\mathbb{P}\left(\|P_M P_\Delta^\perp U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r})UP_\Delta\xi\|_{l^\infty} > \beta\|\xi\|_{l^\infty}\right) \leq \gamma, \tag{7.37}$$

provided that

$$\frac{\beta}{\log\left(\frac{4}{\gamma}(M-s)\right)} \geq C\,\Lambda, \qquad \frac{\beta^2}{\log\left(\frac{4}{\gamma}(M-s)\right)} \geq C\,\Upsilon, \tag{7.38}$$

for some constant $C > 0$, where $q_k = m_k/(N_k - N_{k-1})$ for $k = 1, \ldots, r$,

$$\Lambda = \max_{1\leq k\leq r} \left\{\frac{N_k - N_{k-1}}{m_k} \cdot \left(\sum_{l=1}^{r} \kappa_{\mathbf{N},\mathbf{M}}(k,l)\right)\right\}, \tag{7.39}$$

$$\Upsilon = \max_{1\le l\le r} \sum_{k=1}^{r} \left(\frac{N_k - N_{k-1}}{m_k} - 1 \right) \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot \tilde{s}_k, \tag{7.40}$$

for all $\{\tilde{s}_k\}_{k=1}^r$ such that $\tilde{s}_1 + \ldots + \tilde{s}_r \le s_1 + \ldots + s_r$ and $\tilde{s}_k \le S_k(s_1, \ldots, s_r)$. Moreover, if $q_k = 1$ for all $k = 1, \ldots, r$, then (7.38) is trivially satisfied for any $\gamma > 0$ and the left-hand side of (7.37) is equal to zero.

(ii) If N satisfies the strong Balancing Property with respect to U, M and s, then, for $\xi \in \mathcal{H}$ and $\beta, \gamma > 0$, we have that

$$\mathbb{P}\left(\|P_\Delta^\perp U^*(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r}) U P_\Delta \xi\|_{l^\infty} > \beta \|\xi\|_{l^\infty} \right) \le \gamma, \tag{7.41}$$

provided that

$$\frac{\beta}{\log\left(\frac{4}{\gamma}(\tilde{\theta} - s)\right)} \ge C\,\Lambda, \qquad \frac{\beta^2}{\log\left(\frac{4}{\gamma}(\tilde{\theta} - s)\right)} \ge C\,\Upsilon, \tag{7.42}$$

for some constant $C > 0$, $\tilde{\theta} = \tilde{\theta}(\{q_k\}_{k=1}^r, 1/8, \{N_k\}_{k=1}^r, s, M)$ and Υ, Λ as defined in (i) and

$$\begin{aligned} &\tilde{\theta}(\{q_k\}_{k=1}^r, t, \{N_k\}_{k=1}^r, s, M) \\ &= \left| \left\{ i \in \mathbb{N} : \max_{\substack{\Gamma_1 \subset \{1,\ldots,M\}, \quad |\Gamma_1| = s \\ \Gamma_{2,j} \subset \{N_{j-1}+1,\ldots,N_j\}, \quad j=1,\ldots,r}} \|P_{\Gamma_1} U^*(q_1^{-1} P_{\Gamma_{2,1}} \oplus \ldots \oplus q_r^{-1} P_{\Gamma_{2,r}}) U e_i\| > \frac{t}{\sqrt{s}} \right\} \right|. \end{aligned}$$

Moreover, if $q_k = 1$ for all $k = 1, \ldots, r$, then (7.42) is trivially satisfied for any $\gamma > 0$ and the left-hand side of (7.41) is equal to zero.

Proof. To prove (i) we note that, without loss of generality, we can assume that $\|\xi\|_{l^\infty} = 1$. Let $\{\delta_j\}_{j=1}^N$ be random Bernoulli variables with $\mathbb{P}(\delta_j = 1) = \tilde{q}_j = q_k$, for $j \in \{N_{k-1}+1, \ldots, N_k\}$ and $1 \le k \le r$. A key observation that will be crucial below is that

$$\begin{aligned} P_\Delta^\perp U^*(q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r}) U P_\Delta \xi &= \sum_{j=1}^N P_\Delta^\perp U^* \tilde{q}_j^{-1} \delta_j (e_j \otimes e_j) U P_\Delta \xi \\ &= \sum_{j=1}^N P_\Delta^\perp U^* (\tilde{q}_j^{-1} \delta_j - 1)(e_j \otimes e_j) U P_\Delta \xi + P_\Delta^\perp U^* P_N U P_\Delta \xi. \end{aligned} \tag{7.43}$$

We will use this equation at the end of the argument, but first we will estimate the size of the individual components of $\sum_{j=1}^N P_\Delta^\perp U^*(\tilde{q}_j^{-1}\delta_j - 1)(e_j \otimes e_j) U P_\Delta \xi$. To do that define, for $1 \le j \le N$, the random variables

$$X_j^i = \langle U^*(\tilde{q}_j^{-1}\delta_j - 1)(e_j \otimes e_j) U P_\Delta \xi, e_i \rangle, \qquad i \in \Delta^c.$$

We will show using Bernstein's inequality that, for each $i \in \Delta^c$ and $t > 0$,

$$\mathbb{P}\left(\left| \sum_{j=1}^N X_j^i \right| > t \right) \le 4 \exp\left(-\frac{t^2/4}{\Upsilon + \Lambda t/3} \right). \tag{7.44}$$

To prove the claim, we need to estimate $\mathbb{E}\left(|X_j^i|^2\right)$ and $|X_j^i|$. First note that,

$$\mathbb{E}\left(|X_j^i|^2\right) = (\tilde{q}_j^{-1} - 1) |\langle e_j, U P_\Delta \xi \rangle|^2 |\langle e_j, U e_i \rangle|^2,$$

and note that $|\langle e_j, U e_i \rangle|^2 \le \mu_{\mathbf{N},\mathbf{M}}(k,l)$ for $j \in \{N_{k-1}+1, \ldots, N_k\}$ and $i \in \{M_{l-1}+1, \ldots, M_l\}$. Hence

$$\begin{aligned} \sum_{j=1}^N \mathbb{E}\left(|X_j^i|^2\right) &\le \sum_{k=1}^r (q_k^{-1} - 1) \mu_{\mathbf{N},\mathbf{M}}(k,l) \|P_{N_k}^{N_{k-1}} U P_\Delta \xi\|^2 \\ &\le \sup_{\zeta \in \Theta} \left\{ \sum_{k=1}^r (q_k^{-1} - 1) \mu_{\mathbf{N},\mathbf{M}}(k,l) \|P_{N_k}^{N_{k-1}} U \zeta\|^2 \right\}, \end{aligned}$$

where

$$\Theta = \{\eta : \|\eta\|_{l^\infty} \leq 1, |\mathrm{supp}(P_{M_l}^{M_{l-1}}\eta)| = s_l,\, l = 1, \ldots, r\}.$$

The supremum in the above bound is attained for some $\tilde{\zeta} \in \Theta$. If $\tilde{s}_k = \|P_{N_k}^{N_{k-1}} U\tilde{\zeta}\|^2$, then we have

$$\sum_{j=1}^{N} \mathbb{E}\left(|X_j^i|^2\right) \leq \sum_{k=1}^{r} (q_k^{-1} - 1)\mu_{\mathbf{N},\mathbf{M}}(k,l)\tilde{s}_k. \tag{7.45}$$

Note that it is clear from the definition that $s_k \leq S_k(s_1, \ldots, s_r)$ for $1 \leq k \leq r$. Also, using the fact that $\|U\| \leq 1$ and the definition of Θ, we note that

$$\tilde{s}_1 + \ldots + \tilde{s}_r = \sum_{k=1}^{r} \|P_{N_k}^{N_{k-1}} U P_\Delta \zeta\|^2 \leq \|U P_\Delta \zeta\|^2 = \|\zeta\|^2 \leq s_1 + \ldots + s_r.$$

To estimate $|X_j^i|$ we start by observing that, by the triangle inequality, the fact that $\|\xi\|_{l^\infty} = 1$ and Holder's inequality, it follows that $|\langle \xi, P_\Delta U^* e_j\rangle| \leq \sum_{k=1}^{r} |\langle P_{M_k}^{M_{k-1}}\xi, P_\Delta U^* e_j\rangle|$, and

$$|\langle P_{M_k}^{M_{k-1}}\xi, P_\Delta U^* e_j\rangle| \leq \|P_{N_l}^{N_{l-1}} U P_{\Delta_k}\|_{l^\infty \to l^\infty}, \quad j \in \{N_{l-1}+1, \ldots, N_l\}, \quad l \in \{1, \ldots, r\}.$$

Hence, it follows that for $1 \leq j \leq N$ and $i \in \Delta^c$,

$$\begin{aligned} |X_j^i| &= \tilde{q}_j^{-1}|(\delta_j - \tilde{q}_j)||\langle \xi, P_\Delta U^* e_j\rangle||\langle e_j, U e_i\rangle|, \\ &\leq \max_{1 \leq k \leq r} \left\{ \frac{N_k - N_{k-1}}{m_k} \cdot (\kappa_{\mathbf{N},\mathbf{M}}(k,1) + \ldots + \kappa_{\mathbf{N},\mathbf{M}}(k,r)) \right\}. \end{aligned} \tag{7.46}$$

Now, clearly $\mathbb{E}(X_j^i) = 0$ for $1 \leq j \leq N$ and $i \in \Delta^c$. Thus, by applying Bernstein's inequality to $\mathrm{Re}(X_j^i)$ and $\mathrm{Im}(X_j^i)$ for $j = 1, \ldots, N$, via (7.45) and (7.46), the claim (7.44) follows.

Now, by (7.44), (7.43) and the assumed weak Balancing property (wBP), it follows that

$$\begin{aligned} &\mathbb{P}\left(\|P_M P_\Delta^\perp U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r}) U P_\Delta \xi\|_{l^\infty} > \beta\right) \\ &\leq \sum_{i \in \Delta^c \cap \{1,\ldots,M\}} \mathbb{P}\left(\left| \sum_{j=1}^{N} X_j^i + \langle P_M P_\Delta^\perp U^* P_N^\perp U P_\Delta \xi, e_i\rangle \right| > \beta \right) \\ &\leq \sum_{i \in \Delta^c \cap \{1,\ldots,M\}} \mathbb{P}\left(\left| \sum_{j=1}^{N} X_j^i \right| > \beta - \|P_M P_\Delta^\perp U^* P_N U P_\Delta\|_{l^\infty} \right) \\ &\leq 4(M-s)\exp\left(-\frac{t^2/4}{\Upsilon + \Lambda t/3}\right), \quad t = \frac{1}{2}\beta, \qquad \text{by (7.44), (wBP)}, \end{aligned}$$

Also,

$$4(M-s)\exp\left(-\frac{t^2/4}{\Upsilon + \Lambda t/3}\right) \leq \gamma$$

when

$$\log\left(\frac{4}{\gamma}(M-s)\right)^{-1} \geq \left(\frac{4\Upsilon}{t^2} + \frac{4\Lambda}{3t}\right).$$

And this concludes the proof of (i). To prove (ii), for $t > 0$, suppose that there is a set $\Lambda_t \subset \mathbb{N}$ such that

$$\mathbb{P}\left(\sup_{i \in \Lambda_t} |\langle P_\Delta^\perp U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r}) U P_\Delta \eta, e_i\rangle| > t\right) = 0, \qquad |\Lambda_t^c| < \infty.$$

Then, as before, by (7.44), (7.43) and the assumed strong Balancing property (sBP), it follows that

$$\begin{aligned} &\mathbb{P}\left(\|P_\Delta^\perp U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r}) U P_\Delta \xi\|_{l^\infty} > \beta\right) \\ &\leq \sum_{i \in \Delta^c \cap \Lambda_t^c} \mathbb{P}\left(\left| \sum_{j=1}^{N} X_j^i + \langle P_\Delta^\perp U^* P_N^\perp U P_\Delta \xi, e_i\rangle \right| > \beta \right), \end{aligned}$$

yielding

$$\begin{aligned}
&\mathbb{P}\left(\|P_\Delta^\perp U^*(q_1^{-1}P_{\Omega_1}\oplus\ldots\oplus q_r^{-1}P_{\Omega_r})UP_\Delta\xi\|_{l^\infty}>\beta\right)\\
&\leq \sum_{i\in\Delta^c\cap\Lambda_t^c}\mathbb{P}\left(\left|\sum_{j=1}^N X_j^i\right|>\beta-\|P_\Delta^\perp U^*P_NUP_\Delta\|_{l^\infty}\right)\\
&\leq 4(|\Lambda_t^c|-s)\exp\left(-\frac{t^2/4}{\Upsilon+\Lambda t/3}\right)<\gamma,\quad t=\frac{1}{2}\beta,\qquad \text{by (7.44), (sBP)},
\end{aligned}$$

whenever

$$\log\left(\frac{4}{\gamma}(|\Lambda_t^c|-s)\right)^{-1}\geq\left(\frac{4\Upsilon}{t^2}+\frac{4\Lambda}{3t}\right).$$

Hence, it remains to obtain a bound on $|\Lambda_t^c|$. Let

$$\theta(q_1,\ldots,q_r,t,s)=\left\{i\in\mathbb{N}:\max_{\substack{\Gamma_1\subset\{1,\ldots,M\},\quad|\Gamma_1|=s\\ \Gamma_{2,j}\subset\{N_{j-1}+1,\ldots,N_j\},\quad j=1,\ldots,r}}\|P_{\Gamma_1}U^*(q_1^{-1}P_{\Gamma_{2,1}}\oplus\ldots\oplus q_r^{-1}P_{\Gamma_{2,r}})Ue_i\|>\frac{t}{\sqrt{s}}\right\}.$$

Clearly, $\Delta_t^c\subset\theta(q_1,\ldots,q_r,t,s)$ and

$$\|P_{\Gamma_1}U^*(q_1^{-1}P_{\Gamma_{2,1}}\oplus\ldots\oplus q_r^{-1}P_{\Gamma_{2,r}})Ue_i\|\leq\max_{1\leq j\leq r}q_j^{-1}\|P_NUP_{i-1}^\perp\|\to 0$$

as $i\to\infty$. So, $|\theta(q_1,\ldots,q_r,t,s)|<\infty$. Furthermore, since $\tilde{\theta}(\{q_k\}_{k=1}^r,t,\{N_k\}_{k=1}^r,s,M)$ is a decreasing function in t, for all $t\geq\frac{1}{8}$,

$$|\theta(q_1,\ldots,q_r,t,s)|<\tilde{\theta}(\{q_k\}_{k=1}^r,1/8,\{N_k\}_{k=1}^r,s,M)$$

thus, we have proved (ii). The statements at the end of (i) and (ii) are clear from the reasoning above. □

Proposition 7.12. *Consider the same setup as in Proposition 7.11. If N and K satisfy the weak Balancing Property with respect to U, M and s, then, for $\xi\in\mathcal{H}$ and $\gamma>0$, we have*

$$\mathbb{P}(\|P_\Delta U^*(q_1^{-1}P_{\Omega_1}\oplus\ldots\oplus q_r^{-1}P_{\Omega_r})UP_\Delta-P_\Delta)\xi\|_{l^\infty}>\tilde{\alpha}\|\xi\|_{l^\infty})\leq\gamma,\tag{7.47}$$

$$\tilde{\alpha}=\left(2\log_2^{1/2}\left(4\sqrt{s}KM\right)\right)^{-1},$$

provided that

$$\begin{aligned}
1&\gtrsim\Lambda\cdot\left(\log\left(s\gamma^{-1}\right)+1\right)\cdot\log\left(\sqrt{s}KM\right),\\
1&\gtrsim\Upsilon\cdot\left(\log\left(s\gamma^{-1}\right)+1\right)\cdot\log\left(\sqrt{s}KM\right),
\end{aligned}$$

where Λ and Υ are defined in (7.39) and (7.40). Also,

$$\mathbb{P}(\|P_\Delta U^*(q_1^{-1}P_{\Omega_1}\oplus\ldots\oplus q_r^{-1}P_{\Omega_r})UP_\Delta-P_\Delta)\xi\|_{l^\infty}>\frac{1}{2}\|\xi\|_{l^\infty})\leq\gamma\tag{7.48}$$

provided that

$$1\gtrsim\Lambda\cdot\left(\log\left(s\gamma^{-1}\right)+1\right),\quad 1\gtrsim\Upsilon\cdot\left(\log\left(s\gamma^{-1}\right)+1\right).$$

Moreover, if $q_k=1$ for all $k=1,\ldots,r$, then the left-hand sides of (7.47) and (7.48) are equal to zero.

Proof. Without loss of generality we may assume that $\|\xi\|_{l^\infty}=1$. Let $\{\delta_j\}_{j=1}^N$ be random Bernoulli variables with $\mathbb{P}(\delta_j=1)=\tilde{q}_j:=q_k$, with $j\in\{N_{k-1}+1,\ldots,N_k\}$ and $1\leq k\leq r$. Let also, for $j\in\mathbb{N}$, $\eta_j=(UP_\Delta)^*e_j$. Then, after observing that

$$P_\Delta U^*(q_1^{-1}P_{\Omega_1}\oplus\ldots\oplus q_r^{-1}P_{\Omega_r})UP_\Delta=\sum_{j=1}^N q_j^{-1}\delta_j\eta_j\otimes\bar{\eta}_j,\quad P_\Delta U^*P_NUP_\Delta=\sum_{j=1}^N\eta_j\otimes\bar{\eta}_j,$$

it follows immediately that

$$P_\Delta U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r})UP_\Delta - P_\Delta = \sum_{j=1}^N (\tilde{q}_j^{-1}\delta_j - 1)\eta_j \otimes \bar{\eta}_j - (P_\Delta U^* P_N U P_\Delta - P_\Delta). \quad (7.49)$$

As in the proof of Proposition 7.11 our goal is to eventually use Bernstein's inequality and the following is therefore a setup for that. Define, for $1 \leq j \leq N$, the random variables $Z_j^i = \langle (\tilde{q}_j^{-1}\delta_j - 1)(\eta_j \otimes \bar{\eta}_j)\xi, e_i\rangle$, for $i \in \Delta$. We claim that, for $t > 0$,

$$\mathbb{P}\left(\left|\sum_{j=1}^N Z_j^i\right| > t\right) \leq 4\exp\left(-\frac{t^2/4}{\Upsilon + \Lambda t/3}\right), \qquad i \in \Delta. \quad (7.50)$$

Now, clearly $\mathbb{E}(Z_j^i) = 0$, so we may use Bernstein's inequality. Thus, we need to estimate $\mathbb{E}\left(|Z_j^i|^2\right)$ and $|Z_j^i|$. We will start with $\mathbb{E}\left(|Z_j^i|^2\right)$. Note that

$$\mathbb{E}\left(|Z_j^i|^2\right) = (\tilde{q}_j^{-1} - 1)|\langle e_j, UP_\Delta\xi\rangle|^2|\langle e_j, Ue_i\rangle|^2. \quad (7.51)$$

Thus, we can argue exactly as in the proof of Proposition 7.11 and deduce that

$$\sum_{j=1}^N \mathbb{E}\left(|Z_j^i|^2\right) \leq \sum_{k=1}^r (q_k^{-1} - 1)\mu_{N_{k-1}}\tilde{s}_k, \quad (7.52)$$

where $s_k \leq S_k(s_1, \ldots, s_r)$ for $1 \leq k \leq r$ and $\tilde{s}_1 + \ldots + \tilde{s}_r \leq s_1 + \ldots + s_r$. To estimate $|Z_j^i|$ we argue as in the proof of Proposition 7.11 and obtain

$$|Z_j^i| \leq \max_{1\leq k\leq r}\left\{\frac{N_k - N_{k-1}}{m_k} \cdot (\kappa_{\mathbf{N},\mathbf{M}}(k,1) + \ldots + \kappa_{\mathbf{N},\mathbf{M}}(k,r))\right\}. \quad (7.53)$$

Thus, by applying Bernstein's inequality to $\mathrm{Re}(Z_1^i), \ldots, \mathrm{Re}(Z_N^i)$ and $\mathrm{Im}(Z_1^i), \ldots, \mathrm{Im}(Z_N^i)$ we obtain, via (7.52) and (7.53) the estimate (7.50), and we have proved the claim.

Now armed with (7.50) we can deduce that , by (7.43) and the assumed weak Balancing property (wBP), it follows that

$$\begin{aligned}
&\mathbb{P}\left(\|P_\Delta U^*(q_1^{-1}P_{\Omega_1} \oplus \ldots \oplus q_r^{-1}P_{\Omega_r})UP_\Delta - P_\Delta)\xi\|_{l^\infty} > \tilde{\alpha}\right) \\
&\leq \sum_{i\in\Delta}\mathbb{P}\left(\left|\sum_{j=1}^N Z_j^i + \langle (P_\Delta U^* P_N U P_\Delta - P_\Delta)\xi, e_i\rangle\right| > \tilde{\alpha}\right) \\
&\leq \sum_{i\in\Delta}\mathbb{P}\left(\left|\sum_{j=1}^N Z_j^i\right| > \tilde{\alpha} - \|P_M U^* P_N U P_M - P_M\|_{l^1}\right), \\
&\leq 4\,s\exp\left(-\frac{t^2/4}{\Upsilon + \Lambda t/3}\right), \quad t = \tilde{\alpha}, \qquad \text{by (7.50), (wBP).}
\end{aligned} \quad (7.54)$$

Also,

$$4s\exp\left(-\frac{t^2/4}{\Upsilon + \Lambda t/3}\right) \leq \gamma, \quad (7.55)$$

when

$$1 \geq \left(\frac{4\Upsilon}{t^2} + \frac{4}{3t}\Lambda\right) \cdot \log\left(\frac{4s}{\gamma}\right).$$

And this gives the first part of the proposition. Also, the fact that the left hand side of (7.47) is zero when $q_k = 1$ for $1 \leq k \leq r$ is clear from (7.55). Note that (ii) follows by arguing exactly as above and replacing $\tilde{\alpha}$ by $\frac{1}{4}$.

□

Proposition 7.13. *Let $U \in \mathcal{B}(l^2(\mathbb{N}))$ such that $\|U\| \leq 1$. Suppose that $\Omega = \Omega_{\mathbf{N},\mathbf{m}}$ is a multilevel Bernoulli sampling scheme, where $\mathbf{N} = (N_1, \ldots, N_r) \in \mathbb{N}^r$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Consider $(\mathbf{s}, \mathbf{M})$, where $\mathbf{M} = (M_1, \ldots, M_r) \in \mathbb{N}^r$, $M_1 < \ldots < M_r$, and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, and let $\Delta = \Delta_1 \cup \ldots \cup \Delta_r$, where $\Delta_k \subset \{M_{k-1}+1, \ldots, M_k\}$, $|\Delta_k| = s_k$, and $M_0 = 0$. Then, for any $t \in (0,1)$ and $\gamma \in (0,1)$,*

$$\mathbb{P}\left(\max_{i \in \{1,\ldots,M\} \cap \Delta^c} \|P_{\{i\}} U^* (q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r}) U P_{\{i\}}\| \geq 1 + t\right) \leq \gamma$$

provided that

$$\frac{t^2}{4} \geq \log\left(\frac{2M}{\gamma}\right) \cdot \max_{1 \leq k \leq r} \left\{\left(\frac{N_k - N_{k-1}}{m_k} - 1\right) \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l)\right\} \tag{7.56}$$

for all $l = 1, \ldots, r$ when $M = M_r$ and for all $l = 1, \ldots, r-1, \infty$ when $M > M_r$. In addition, if $m_k = N_k - N_{k-1}$ for each $k = 1, \ldots r$, then

$$\mathbb{P}(\|P_{\{i\}} U^* (q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r}) U P_{\{i\}}\| \geq 1 + t) = 0, \quad \forall i \in \mathbb{N}. \tag{7.57}$$

Proof. Fix $i \in \{1, \ldots, M\}$. Let $\{\delta_j\}_{j=1}^N$ be random independent Bernoulli variables with $\mathbb{P}(\delta_j = 1) = \tilde{q}_j := q_k$ for $j \in \{N_{k-1}+1, \ldots, N_k\}$. Define $Z = \sum_{j=1}^N Z_j$ and $Z_j = \left(\tilde{q}_j^{-1} \delta_j - 1\right) |u_{ji}|^2$. Now observe that

$$P_{\{i\}} U^* (q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r}) U P_{\{i\}} = \sum_{j=1}^N \tilde{q}_j^{-1} \delta_j |u_{ji}|^2 = \sum_{j=1}^N Z_j + \sum_{j=1}^N |u_{ji}|^2,$$

where we interpret U as the infinite matrix $U = \{u_{ij}\}_{i,j \in \mathbb{N}}$. Thus, since $\|U\| \leq 1$,

$$\|P_{\{i\}} U^* (q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r}) U P_{\{i\}}\| \leq \left|\sum_{j=1}^N Z_j\right| + 1 \tag{7.58}$$

and it is clear that (7.57) is true. For the case where $q_k < 1$ for some $k \in \{1, \ldots, r\}$, observe that for $i \in \{M_{l-1}+1, \ldots, M_l\}$ (recall that Z_j depend on i), we have that $\mathbb{E}(Z_j) = 0$. Also,

$$|Z_j| \leq \begin{cases} \max_{1 \leq k \leq r}\{\max\{q_k^{-1} - 1, 1\} \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l)\} := B_i & i \in \{M_{l-1}+1, \ldots, M_l\} \\ \max_{1 \leq k \leq r}\{\max\{q_k^{-1} - 1, 1\} \cdot \mu_{\mathbf{N},\mathbf{M}}(k,\infty)\} := B_\infty & i > M_r, \end{cases}$$

and, by again using the assumption that $\|U\| \leq 1$,

$$\begin{aligned} \sum_{j=1}^N \mathbb{E}(|Z_j|^2) &= \sum_{j=1}^N (\tilde{q}_j^{-1} - 1) |u_{ji}|^4 \\ &\leq \begin{cases} \max_{1 \leq k \leq r}\{(q_k^{-1} - 1) \mu_{\mathbf{N},\mathbf{M}}(k,l)\} =: \sigma_i^2 & i \in \{M_{l-1}+1, \ldots, M_l\} \\ \max_{1 \leq k \leq r}\{(q_k^{-1} - 1) \mu_{\mathbf{N},\mathbf{M}}(k,\infty)\} =: \sigma_\infty^2 & i > M_r. \end{cases} \end{aligned}$$

Thus, by Bernstein's inequality and (7.58),

$$\begin{aligned} &\mathbb{P}(\|P_{\{i\}} U^* (q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r}) U P_{\{i\}}\| \geq 1 + t) \\ &\qquad \leq \mathbb{P}\left(\left|\sum_{j=1}^N Z_j\right| \geq t\right) \leq 2 \exp\left(-\frac{t^2/2}{\sigma^2 + Bt/3}\right), \end{aligned}$$

$$B = \begin{cases} \max_{1 \leq i \leq r} B_i & M = M_r, \\ \max_{i \in \{1,\ldots,r-1,\infty\}} B_i & M > M_r \end{cases}, \quad \sigma^2 = \begin{cases} \max_{1 \leq i \leq r} \sigma_i^2 & M = M_r, \\ \max_{i \in \{1,\ldots,r-1,\infty\}} \sigma_1^2 & M > M_r. \end{cases}$$

Applying the union bound yields

$$\mathbb{P}\left(\max_{i \in \{1,\ldots,M\}} \|P_{\{i\}} U^* (q_1^{-1} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1} P_{\Omega_r}) U P_{\{i\}}\| \geq 1 + t\right) \leq \gamma$$

whenever (7.56) holds. □

7.3 Proofs of Propositions 7.3 and 7.4

The proof of the propositions relies on an idea that originated in a paper by D. Gross [39], namely, the golfing scheme. The variant we are using here is based on an idea from [1]. However, the informed reader will recognise that the setup here differs substantially from both [39] and [1]. See also [14] for other examples of the use of the golfing scheme. Before we embark on the proof, we will state and prove a useful lemma.

Lemma 7.14. *Let $\tilde{X}_k$ be independent binary variables taking values 0 and 1, such that $\tilde{X}_k = 1$ with probability P. Then,*

$$\mathbb{P}\left(\sum_{i=1}^{N} \tilde{X}_i \geq k\right) \geq \left(\frac{N \cdot e}{k}\right)^{-k} \binom{N}{k} P^k. \tag{7.59}$$

Proof. First observe that

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{N} \tilde{X}_i \geq k\right) &= \sum_{i=k}^{N} \binom{N}{i} P^i (1-P)^{N-i} = \sum_{i=0}^{N-k} \binom{N}{i+k} P^{i+k} (1-P)^{N-k-i} \\
&= \binom{N}{k} P^k \sum_{i=0}^{N-k} \frac{(N-k)!k!}{(N-i-k)!(i+k)!} P^i (1-P)^{N-k-i} \\
&= \binom{N}{k} P^k \sum_{i=0}^{N-k} \binom{N-k}{i} P^i (1-P)^{N-k-i} \left[\binom{i+k}{k}\right]^{-1}.
\end{aligned}$$

The result now follows because $\sum_{i=0}^{N-k} \binom{N-k}{i} P^i (1-P)^{N-k-i} = 1$ and for $i = 0, \ldots, N-k$, we have that

$$\binom{i+k}{k} \leq \left(\frac{(i+k) \cdot e}{k}\right)^k \leq \left(\frac{N \cdot e}{k}\right)^k,$$

where the first inequality follows from Stirling's approximation (see [19], p. 1186). □

Proof of Proposition 7.3. We start by mentioning that converting from the Bernoulli sampling model and uniform sampling model has become standard in the literature. In particular, one can do this by showing that the Bernoulli model implies (up to a constant) the uniform sampling model in each of the conditions in Proposition 7.1. This is straightforward and the reader may consult [16, 15, 36] for details. We will therefore consider (without loss of generality) only the multilevel Bernoulli sampling scheme.

Recall that we are using the following Bernoulli sampling model: Given $N_0 = 0$, $N_1, \ldots, N_r \in \mathbb{N}$ we let

$$\{N_{k-1}+1, \ldots, N_k\} \supset \Omega_k \sim \mathrm{Ber}\,(q_k)\,, \quad q_k = \frac{m_k}{N_k - N_{k-1}}.$$

Note that we may replace this Bernoulli sampling model with the following equivalent sampling model (see [1]):

$$\Omega_k = \Omega_k^1 \cup \Omega_k^2 \cup \cdots \cup \Omega_k^u, \qquad \Omega_k^j \sim \mathrm{Ber}(q_k^j), \qquad 1 \leq k \leq r,$$

for some $u \in \mathbb{N}$ with

$$(1-q_k^1)(1-q_k^2)\cdots(1-q_k^u) = (1-q_k). \tag{7.60}$$

The latter model is the one we will use throughout the proof and the specific value of u will be chosen later. Note also that because of overlaps we will have

$$q_k^1 + q_k^2 + \ldots + q_k^u \geq q_k, \qquad 1 \leq k \leq r. \tag{7.61}$$

The strategy of the proof is to show the validity of (i) and (ii), and the existence of a $\rho \in \mathrm{ran}(U^*(P_{\Omega_1} \oplus \ldots \oplus P_{\Omega_r}))$ that satisfies (iii)-(v) in Proposition 7.1 with probability exceeding $1-\epsilon$, where (iii) is replaced by (7.16), (iv) is replaced by $\|P_M P_\Delta^\perp \rho\|_{l^\infty} \leq \frac{1}{2}$ and L in (v) is given by (7.17).

Step I: The construction of ρ: We start by defining $\gamma = \epsilon/6$ (the reason for this particular choice will become clear later). We also define a number of quantities (and the reason for these choices will become clear later in the proof):

$$u = 8\lceil 3v + \log(\gamma^{-1})\rceil, \qquad v = \lceil \log_2(8KM\sqrt{s})\rceil, \tag{7.62}$$

as well as

$$\{q_k^i : 1 \le k \le r, 1 \le i \le u\}, \quad \{\alpha_i\}_{i=1}^u, \quad \{\beta_i\}_{i=1}^u$$

by

$$q_k^1 = q_k^2 = \frac{1}{4} q_k, \qquad \tilde{q}_k = q_k^3 = \ldots = q_k^u, \qquad q_k = (N_k - N_{k-1}) m_k^{-1}, \quad 1 \le k \le r, \tag{7.63}$$

with

$$(1 - q_k^1)(1 - q_k^2) \cdots (1 - q_k^u) = (1 - q_k)$$

and

$$\alpha_1 = \alpha_2 = (2 \log_2^{1/2}(4KM\sqrt{s}))^{-1}, \qquad \alpha_i = 1/2, \quad 3 \le i \le u, \tag{7.64}$$

as well as

$$\beta_1 = \beta_2 = \frac{1}{4}, \qquad \beta_i = \frac{1}{4} \log_2(4KM\sqrt{s}), \quad 3 \le i \le u. \tag{7.65}$$

Consider now the following construction of ρ. We will define recursively the sequences $\{Z_i\}_{i=0}^u \subset \mathcal{H}$, $\{Y_i\}_{i=1}^u \subset \mathcal{H}$ and $\{\omega_i\}_{i=0}^u \subset \mathbb{N}$ as follows: first let $\omega_0 = \{0\}$, $\omega_1 = \{0, 1\}$ and $\omega_2 = \{0, 1, 2\}$. Then define recursively, for $i \ge 3$, the following:

$$\omega_i = \begin{cases} \omega_{i-1} \cup \{i\} & \text{if } \|(P_\Delta - P_\Delta U^*(\frac{1}{q_1^i} P_{\Omega_1^i} \oplus \ldots \oplus \frac{1}{q_r^i} P_{\Omega_r^i}) U P_\Delta) Z_{i-1}\|_{l^\infty} \le \alpha_i \|P_{\Delta_k} Z_{i-1}\|_{l^\infty}, \\ & \text{and } \|P_M P_\Delta^\perp U^*(\frac{1}{q_1^i} P_{\Omega_1^i} \oplus \ldots \oplus \frac{1}{q_r^i} P_{\Omega_r^i}) U P_\Delta Z_{i-1}\|_{l^\infty} \le \beta_i \|Z_{i-1}\|_{l^\infty}, \\ \omega_{i-1} & \text{otherwise}, \end{cases}$$

$$Y_i = \begin{cases} \sum_{j \in \omega_i} U^*(\frac{1}{q_1^j} P_{\Omega_1^j} \oplus \ldots \oplus \frac{1}{q_r^j} P_{\Omega_r^j}) U Z_{j-1} & \text{if } i \in \omega_i, \\ Y_{i-1} & \text{otherwise}, \end{cases} \qquad i \ge 1, \tag{7.66}$$

$$Z_i = \begin{cases} \operatorname{sgn}(x_0) - P_\Delta Y_i & \text{if } i \in \omega_i, \\ Z_{i-1} & \text{otherwise}, \end{cases} \qquad i \ge 1, \qquad Z_0 = \operatorname{sgn}(x_0).$$

Now, let $\{A_i\}_{i=1}^2$ and $\{B_i\}_{i=1}^5$ denote the following events

$$\begin{aligned} A_i : &\quad \|(P_\Delta - U^*(\frac{1}{q_1^i} P_{\Omega_1^i} \oplus \ldots \oplus \frac{1}{q_r^i} P_{\Omega_r^i}) U P_\Delta) Z_{i-1}\|_{l^\infty} \le \alpha_i \|Z_{i-1}\|_{l^\infty}, \qquad i = 1, 2, \\ B_i : &\quad \|P_M P_\Delta^\perp U^*(\frac{1}{q_1^i} P_{\Omega_1^i} \oplus \ldots \oplus \frac{1}{q_r^i} P_{\Omega_r^i}) U P_\Delta Z_{i-1}\|_{l^\infty} \le \beta_i \|Z_{i-1}\|_{l^\infty}, \qquad i = 1, 2, \\ B_3 : &\quad \|P_\Delta U^*(\frac{1}{q_1} P_{\Omega_1} \oplus \ldots \oplus \frac{1}{q_r} P_{\Omega_r}) U P_\Delta - P_\Delta\| \le 1/4, \\ &\quad \max_{i \in \Delta^c \cap \{1, \ldots, M\}} \| \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} P_{\Omega_r} \right) U e_i \| \le \sqrt{5/4} \\ B_4 : &\quad |\omega_u| \ge v, \\ B_5 : &\quad (\cap_{i=1}^2 A_i) \cap (\cap_{i=1}^4 B_i). \end{aligned} \tag{7.67}$$

Also, let $\tau(j)$ denote the jth element in ω_u (e.g. $\tau(0) = 0, \tau(1) = 1, \tau(2) = 2$ etc.) and finally define ρ by

$$\rho = \begin{cases} Y_{\tau(v)} & \text{if } B_5 \text{ occurs}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that, clearly, $\rho \in \operatorname{ran}(U^* P_\Omega)$, and we just need to show that when the event B_5 occurs, then (i)-(v) in Proposition 7.1 will follow.

Step II: $B_5 \Rightarrow$ **(i), (ii).** To see that the assertion is true, note that if B_5 occurs then B_3 occurs, which immediately (i) and (ii).

Step III: $B_5 \Rightarrow$ (**iii**), (**iv**). To show the assertion, we start by making the following observations: By the construction of $Z_{\tau(i)}$ and the fact that $Z_0 = \mathrm{sgn}(x_0)$, it follows that

$$\begin{aligned} Z_{\tau(i)} &= Z_0 - (P_\Delta U^*(\frac{1}{q_1^{\tau(1)}} P_{\Omega_1^{\tau(1)}} \oplus \ldots \oplus \frac{1}{q_r^{\tau(1)}} P_{\Omega_r^{\tau(i)}}) U P_\Delta) Z_0 \\ &\quad + \ldots + P_\Delta U^*(\frac{1}{q_1^{\tau(i)}} P_{\Omega_1^{\tau(i)}} \oplus \ldots \oplus \frac{1}{q_r^{\tau(i)}} P_{\Omega_r^{\tau(i)}}) U P_\Delta) Z_{\tau(i-1)}) \\ &= Z_{\tau(i-1)} - P_\Delta U^*(\frac{1}{q_1^{\tau(i)}} P_{\Omega_1^{\tau(i)}} \oplus \ldots \oplus \frac{1}{q_r^{\tau(i)}} P_{\Omega_r^{\tau(i)}}) U P_\Delta) Z_{\tau(i-1)} \qquad i \le |\omega_u|, \end{aligned}$$

so we immediately get that

$$Z_{\tau(i)} = (P_\Delta - P_\Delta U^*(\frac{1}{q_1^{\tau(i)}} P_{\Omega_1^{\tau(i)}} \oplus \ldots \oplus \frac{1}{q_r^{\tau(i)}} P_{\Omega_r^{\tau(i)}}) U P_\Delta) Z_{\tau(i-1)}, \qquad i \le |\omega_u|.$$

Hence, if the event B_5 occurs, we have, by the choices in (7.64) and (7.65)

$$\|\rho - \mathrm{sgn}(x_0)\| = \|Z_{\tau(v)}\| \le \sqrt{s}\|Z_{\tau(v)}\|_{l^\infty} \le \sqrt{s}\prod_{i=1}^{v} \alpha_{\tau(i)} \le \frac{\sqrt{s}}{2^v} \le \frac{1}{8K}, \tag{7.68}$$

since we have chosen $v = \lceil \log_2(8KM\sqrt{s}) \rceil$. Also,

$$\begin{aligned} \|P_M P_\Delta^\perp \rho\|_{l^\infty} &\le \sum_{i=1}^{v} \|P_M P_\Delta^\perp U^*(\frac{1}{q_1^{\tau(i)}} P_{\Omega_1^{\tau(i)}} \oplus \ldots \oplus \frac{1}{q_r^{\tau(i)}} P_{\Omega_r^{\tau(i)}}) U P_\Delta Z_{\tau(i-1)}\|_{l^\infty} \\ &\le \sum_{i=1}^{v} \beta_{\tau(i)} \|Z_{\tau(i-1)}\|_{l^\infty} \le \sum_{i=1}^{v} \beta_{\tau(i)} \prod_{j=1}^{i-1} \alpha_{\tau(j)} \\ &\le \frac{1}{4}(1 + \frac{1}{2\log_2^{1/2}(a)} + \frac{\log_2(a)}{2^3 \log_2(a)} + \ldots + \frac{1}{2^{v-1}}) \le \frac{1}{2}, \qquad a = 4KM\sqrt{s}. \end{aligned} \tag{7.69}$$

In particular, (7.68) and (7.69) imply (iii) and (iv) in Proposition 7.1.

Step IV: $B_5 \Rightarrow$ (**v**). To show that, note that we may write the already constructed ρ as $\rho = U^* P_\Omega w$ where

$$w = \sum_{i=1}^{v} w_i, \quad w_i = \left(\frac{1}{q_1^{\tau(i)}} P_{\Omega_1} \oplus \ldots \oplus \frac{1}{q_r^{\tau(i)}} P_{\Omega_r}\right) U P_\Delta Z_{\tau(i-1)}.$$

To estimate $\|w\|$ we simply compute

$$\begin{aligned} \|w_i\|^2 &= \left\langle \left(\frac{1}{q_1^{\tau(i)}} P_{\Omega_1^{\tau(i)}} \oplus \ldots \oplus \frac{1}{q_r^{\tau(i)}} P_{\Omega_r^{\tau(i)}}\right) U P_\Delta Z_{\tau(i-1)}, \left(\frac{1}{q_1^{\tau(i)}} P_{\Omega_1^{\tau(i)}} \oplus \ldots \oplus \frac{1}{q_r^{\tau(i)}} P_{\Omega_r^{\tau(i)}}\right) U P_\Delta Z_{\tau(i-1)} \right\rangle \\ &= \sum_{k=1}^{r} \left(\frac{1}{q_k^{\tau(i)}}\right)^2 \|P_{\Omega_k^{\tau(i)}} U Z_{\tau(i-1)}\|^2, \end{aligned}$$

and then use the assumption that the event B_5 holds to deduce that

$$\begin{aligned} &\sum_{k=1}^{r} \left(\frac{1}{q_k^{\tau(i)}}\right)^2 \|P_{\Omega_k^{\tau(i)}} U Z_{\tau(i-1)}\|^2 \le \max_{1\le k\le r} \left\{\frac{1}{q_k^{\tau(i)}}\right\} \langle \sum_{k=1}^{r} \frac{1}{q_k^{\tau(i)}} P_\Delta U^* P_{\Omega_k^{\tau(i)}} U Z_{\tau(i-1)}, Z_{\tau(i-1)} \rangle \\ &= \max_{1\le k\le r} \left\{\frac{1}{q_k^{\tau(i)}}\right\} \langle \left(\sum_{k=1}^{r} \frac{1}{q_k^{\tau(i)}} P_\Delta U^* P_{\Omega_k^{\tau(i)}} U - P_\Delta\right) Z_{\tau(i-1)}, Z_{\tau(i-1)} \rangle + \|Z_{\tau(i-1)}\|^2 \\ &\le \max_{1\le k\le r} \left\{\frac{1}{q_k^{\tau(i)}}\right\} \left(\|Z_{\tau(i-1)}\| \|Z_{\tau(i)}\| + \|Z_{\tau(i-1)}\|^2\right) \\ &\le \max_{1\le k\le r} \left\{\frac{1}{q_k^{\tau(i)}}\right\} s\left(\|Z_{\tau(i-1)}\|_{l^\infty} \|Z_{\tau(i)}\|_{l^\infty} + \|Z_{\tau(i-1)}\|_{l^\infty}^2\right) \le \max_{1\le k\le r} \left\{\frac{1}{q_k^{\tau(i)}}\right\} s(\alpha_i + 1) \left(\prod_{j=1}^{i-1} \alpha_j\right)^2, \end{aligned}$$

where the last inequality follows from the assumption that the event B_5 holds. Hence

$$\|w\| \leq \sqrt{s}\sum_{i=1}^{v}\left(\max_{1\leq k\leq r}\left\{\frac{1}{\sqrt{q_k^{\tau(i)}}}\right\}\sqrt{\alpha_i+1}\prod_{j=1}^{i-1}\alpha_j\right) \tag{7.70}$$

Note that, due to the fact that $q_k^1+\ldots+q_k^u \geq q_k$, we have that

$$\tilde{q}_k \geq \frac{m_k}{2(N_k-N_{k-1})}\frac{1}{8\left\lceil\log(\gamma^{-1})+3\lceil\log_2(8KM\sqrt{s})\rceil\right\rceil-2}.$$

This gives, in combination with the chosen values of $\{\alpha_j\}$ and (7.70) that

$$\begin{aligned}
\|w\| &\leq 2\sqrt{s}\max_{1\leq k\leq r}\sqrt{\frac{N_k-N_{k-1}}{m_k}}\left(1+\frac{1}{2\log_2^{1/2}(4KM\sqrt{s})}\right)^{3/2} \\
&\quad+\sqrt{s}\max_{1\leq k\leq r}\sqrt{\frac{N_k-N_{k-1}}{m_k}}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{8\left\lceil\log(\gamma^{-1})+3\lceil\log_2(8KM\sqrt{s})\rceil\right\rceil-2}}{\log_2(4KM\sqrt{s})}\cdot\sum_{i=3}^{v}\frac{1}{2^{i-3}} \\
&\leq 2\sqrt{s}\max_{1\leq k\leq r}\sqrt{\frac{N_k-N_{k-1}}{m_k}}\left(\left(\frac{3}{2}\right)^{3/2}+\sqrt{\frac{6}{\log_2(4KM\sqrt{s})}}\sqrt{1+\frac{\log_2(\gamma^{-1})+6}{\log_2(4KM\sqrt{s})}}\right) \\
&\leq \sqrt{s}\max_{1\leq k\leq r}\sqrt{\frac{N_k-N_{k-1}}{m_k}}\left(\frac{3\sqrt{3}}{\sqrt{2}}+\frac{2\sqrt{6}}{\sqrt{\log_2(4KM\sqrt{s})}}\sqrt{1+\frac{\log_2(\gamma^{-1})+6}{\log_2(4KM\sqrt{s})}}\right).
\end{aligned} \tag{7.71}$$

Step V: The weak balancing property, (7.14) and (7.15) $\Rightarrow \mathbb{P}(A_1^c\cup A_2^c\cup B_1^c\cup B_2^c\cup B_3^c)\leq 5\gamma$.

To see this, note that by Proposition 7.12 we immediately get (recall that $q_k^1=q_k^2=1/4q_k$) that $\mathbb{P}(A_1^c)\leq\gamma$ and $\mathbb{P}(A_2^c)\leq\gamma$ as long as the weak balancing property and

$$1\gtrsim\Lambda\cdot\left(\log\left(s\gamma^{-1}\right)+1\right)\cdot\log\left(\sqrt{s}KM\right),\quad 1\gtrsim\Upsilon\cdot\left(\log\left(s\gamma^{-1}\right)+1\right)\cdot\log\left(\sqrt{s}KM\right), \tag{7.72}$$

are satisfied, where $K=\max_{1\leq k\leq r}(N_k-N_{k-1})/m_k$,

$$\Lambda=\max_{1\leq k\leq r}\left\{\frac{N_k-N_{k-1}}{m_k}\cdot\left(\sum_{l=1}^{r}\kappa_{\mathbf{N},\mathbf{M}}(k,l)\right)\right\}, \tag{7.73}$$

$$\Upsilon=\max_{1\leq l\leq r}\sum_{k=1}^{r}\left(\frac{N_k-N_{k-1}}{m_k}-1\right)\cdot\mu_{\mathbf{N},\mathbf{M}}(k,l)\cdot\tilde{s}_k, \tag{7.74}$$

and where $\tilde{s}_1+\ldots+\tilde{s}_r\leq s_1+\ldots+s_r$ and $\tilde{s}_k\leq S_k(s_1,\ldots,s_r)$. However, clearly, (7.14) and (7.15) imply (7.72). Also, Proposition 7.11 yields that $\mathbb{P}(B_1^c)\leq\gamma$ and $\mathbb{P}(B_2^c)\leq\gamma$ as long as the weak balancing property and

$$1\gtrsim\Lambda\cdot\log\left(\frac{4}{\gamma}(M-s)\right),\qquad 1\gtrsim\Upsilon\cdot\log\left(\frac{4}{\gamma}(M-s)\right), \tag{7.75}$$

are satisfied. However, again, (7.14) and (7.15) imply (7.75). Finally, it remains to bound $\mathbb{P}(B_3^c)$. First note that by Theorem 7.8, we may deduce that

$$\mathbb{P}\left(\|P_\Delta U^*(\frac{1}{q_1}P_{\Omega_1}\oplus\ldots\oplus\frac{1}{q_r}P_{\Omega_r})UP_\Delta-P_\Delta\|>1/4,\right)\leq\gamma/2,$$

when the weak balancing property and

$$1\gtrsim\Lambda\cdot\left(\log\left(\gamma^{-1}s\right)+1\right) \tag{7.76}$$

holds and (7.14) implies (7.76).

For the second part of B_3, we may deduce from Proposition 7.13 that

$$\mathbb{P}\left(\max_{i\in\Delta^c\cap\{1,\ldots,M\}}\|\left(q_1^{-1/2}P_{\Omega_1}\oplus\ldots\oplus q_r^{-1/2}P_{\Omega_r}\right)Ue_i\|>\sqrt{5/4}\right)\leq\frac{\gamma}{2},$$

whenever

$$1 \gtrsim \log\left(\frac{2M}{\gamma}\right) \cdot \max_{1\leq k\leq r}\left\{\left(\frac{N_k - N_{k-1}}{m_k} - 1\right)\cdot \mu_{\mathbf{N},\mathbf{M}}(k,l)\right\}, \qquad l = 1,\ldots,r. \tag{7.77}$$

which is true whenever (7.14) holds. Indeed, recalling the definition of $\kappa_{\mathbf{N},\mathbf{M}}(k,j)$ and Θ in Definition 7.2, observe that

$$\max_{\eta\in\Theta,\|\eta\|_\infty=1}\sum_{l=1}^{r}\left\|P_{N_k}^{N_{k-1}}UP_{M_l}^{M_{l-1}}\eta\right\|_\infty \geq \max_{\eta\in\Theta,\|\eta\|_\infty=1}\left\|P_{N_k}^{N_{k-1}}U\eta\right\|_\infty \geq \sqrt{\mu(P_{N_k}^{N_{k-1}}UP_{M_l}^{M_{l-1}})} \tag{7.78}$$

for each $l = 1,\ldots,r$ which implies that $\sum_{j=1}^{r}\kappa_{\mathbf{N},\mathbf{M}}(k,j) \geq \mu_{\mathbf{N},\mathbf{M}}(k,l)$, for $l = 1,\ldots,r$. Consequently, (7.77) follows from (7.14). Thus, $\mathbb{P}(B_3^c) \leq \gamma$.

Step VI: The weak balancing property, (7.14) and (7.15) $\Rightarrow$ $\mathbb{P}(B_4^c) \leq \gamma$. To see this, define the random variables $X_1,\ldots X_{u-2}$ by

$$X_j = \begin{cases} 0 & \omega_{j+2} \neq \omega_{j+1}, \\ 1 & \omega_{j+2} = \omega_{j+1}. \end{cases} \tag{7.79}$$

We immediately observe that

$$\mathbb{P}(B_4^c) = \mathbb{P}(|\omega_u| < v) = \mathbb{P}(X_1 + \ldots + X_{u-2} > u - v). \tag{7.80}$$

However, the random variables $X_1,\ldots X_{u-2}$ are not independent, and we therefore cannot directly apply the standard Chernoff bound. In particular, we must adapt the setup slightly. Note that

$$\begin{aligned}
&\mathbb{P}(X_1 + \ldots + X_{u-2} > u - v) \\
&\leq \sum_{l=1}^{\binom{u-2}{u-v}} \mathbb{P}(X_{\pi(l)_1} = 1, X_{\pi(l)_2} = 1, \ldots, X_{\pi(l)_{u-v}} = 1) \\
&= \sum_{l=1}^{\binom{u-2}{u-v}} \mathbb{P}(X_{\pi(l)_{u-v}} = 1 \,|\, X_{\pi(l)_1} = 1, \ldots, X_{\pi(l)_{u-v-1}} = 1)\mathbb{P}(X_{\pi(l)_1} = 1, \ldots, X_{\pi(l)_{u-v-1}} = 1) \\
&= \sum_{l=1}^{\binom{u-2}{u-v}} \mathbb{P}(X_{\pi(l)_{u-v}} = 1 \,|\, X_{\pi(l)_1} = 1, \ldots, X_{\pi(l)_{u-v-1}} = 1) \\
&\qquad \times \mathbb{P}(X_{\pi(l)_{u-v-1}} = 1 \,|\, X_{\pi(l)_1} = 1, \ldots, X_{\pi(l)_{u-v-2}} = 1) \cdots \mathbb{P}(X_{\pi(l)_1} = 1)
\end{aligned} \tag{7.81}$$

where $\pi : \{1,\ldots,\binom{u-2}{u-v}\} \to \mathbb{N}^{u-v}$ ranges over all $\binom{u-2}{u-v}$ ordered subsets of $\{1,\ldots,u-2\}$ of size $u - v$. Thus, if we can provide a bound P such that

$$\begin{aligned}
&P \geq \mathbb{P}(X_{\pi(l)_{u-v-j}} = 1 \,|\, X_{\pi(l)_1} = 1, \ldots, X_{\pi(l)_{u-v-(j+1)}} = 1), \\
&P \geq \mathbb{P}(X_{\pi(l)_1} = 1)
\end{aligned} \tag{7.82}$$

$$l = 1,\ldots,\binom{u-2}{u-v}, \quad j = 0,\ldots,u-v-2,$$

then, by (7.81),

$$\mathbb{P}(X_1 + \ldots + X_{u-2} > u - v) \leq \binom{u-2}{u-v}P^{u-v}. \tag{7.83}$$

We will continue assuming that (7.82) is true, and then return to this inequality below.

Let $\{\tilde{X}_k\}_{k=1}^{u-2}$ be independent binary variables taking values 0 and 1, such that $\tilde{X}_k = 1$ with probability P. Then, by Lemma 7.14, (7.83) and (7.80) it follows that

$$\mathbb{P}(B_4^c) \leq \mathbb{P}\left(\tilde{X}_1 + \ldots + \tilde{X}_{u-2} \geq u - v\right)\left(\frac{(u-2)\cdot e}{u-v}\right)^{u-v}. \tag{7.84}$$

Then, by the standard Chernoff bound ([57, Theorem 2.1, equation 2]), it follows that, for $t > 0$,

$$\mathbb{P}\left(\tilde{X}_1 + \ldots + \tilde{X}_{u-2} \geq (u-2)(t+P)\right) \leq e^{-2(u-2)t^2}. \tag{7.85}$$

Hence, if we let $t = (u-v)/(u-2) - P$, it follows from (7.84) and (7.85) that

$$\mathbb{P}(B_4^c) \leq e^{-2(u-2)t^2 + (u-v)(\log(\frac{u-2}{u-v})+1)} \leq e^{-2(u-2)t^2+u-2}.$$

Thus, by choosing $P = 1/4$ we get that $\mathbb{P}(B_4^c) \leq \gamma$ whenever $u \geq x$ and x is the largest root satisfying

$$(x-u)\left(\frac{x-v}{u-2} - \frac{1}{4}\right) - \log(\gamma^{-1/2}) - \frac{x-2}{2} = 0,$$

and this yields $u \geq 8\lceil 3v + \log(\gamma^{-1/2})\rceil$ which is satisfied by the choice of u in (7.62). Thus, we would have been done with Step VI if we could verify (7.82) with $P = 1/4$, and this is the theme in the following claim.

Claim: The weak balancing property, (7.14) and (7.15) $\Rightarrow$ (7.82) with $P = 1/4$. To prove the claim we first observe that $X_j = 0$ when

$$\|(P_\Delta - P_\Delta U^*(\frac{1}{q_1^i}P_{\Omega_1^i} \oplus \ldots \oplus \frac{1}{q_r^i}P_{\Omega_r^i})UP_\Delta)Z_{i-1}\|_{l^\infty} \leq \frac{1}{2}\|Z_{i-1}\|_{l^\infty}$$

$$\|P_M P_\Delta^\perp U^*(\frac{1}{q_1^i}P_{\Omega_1^i} \oplus \ldots \oplus \frac{1}{q_r^i}P_{\Omega_r^i})UP_\Delta Z_{i-1}\|_{l^\infty} \leq \frac{1}{4}\log_2(4KM\sqrt{s})\|Z_{i-1}\|_{l^\infty}, \qquad i = j+2,$$

where we recall from (7.63) that

$$q_k^3 = q_k^4 = \ldots = q_k^u = \tilde{q}_k, \qquad 1 \leq k \leq r.$$

Thus, by choosing $\gamma = 1/8$ in (7.48) in Proposition 7.12 and $\gamma = 1/8$ in (i) in Proposition 7.11, it follows that $\frac{1}{4} \geq \mathbb{P}(X_j = 1)$, for $j = 1, \ldots, u-2$, when the weak balancing property is satisfied and

$$(\log(8s)+1)^{-1} \gtrsim \tilde{q}_k^{-1} \cdot \sum_{l=1}^{r} \kappa_{\mathbf{N},\mathbf{M}}(k,l), \quad 1 \leq k \leq r \tag{7.86}$$

$$(\log(8s)+1)^{-1} \gtrsim \left(\sum_{k=1}^{r} \left(\tilde{q}_k^{-1} - 1\right) \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot \tilde{s}_k\right), \quad 1 \leq l \leq r, \tag{7.87}$$

as well as

$$\frac{\log_2(4KM\sqrt{s})}{\log(32(M-s))} \gtrsim \tilde{q}_k^{-1} \cdot \sum_{l=1}^{r} \kappa_{\mathbf{N},\mathbf{M}}(k,l), \quad 1 \leq k \leq r \tag{7.88}$$

$$\frac{\log_2(4KM\sqrt{s})}{\log(32(M-s))} \gtrsim \left(\sum_{k=1}^{r} \left(\tilde{q}_k^{-1} - 1\right) \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot \tilde{s}_k\right), \quad 1 \leq l \leq r, \tag{7.89}$$

with $K = \max_{1\leq k\leq r}(N_k - N_{k-1})/m_k$. Thus, to prove the claim we must demonstrate that (7.14) and (7.15) $\Rightarrow$ (7.86), (7.87), (7.88) and (7.89). We split this into two stages:

Stage 1: (7.15) $\Rightarrow$ (7.89) and (7.87). To show the assertion we must demonstrate that if, for $1 \leq k \leq r$,

$$m_k \gtrsim (\log(s\epsilon^{-1})+1) \cdot \hat{m}_k \cdot \log\left(KM\sqrt{s}\right), \tag{7.90}$$

where $\hat{m}_k$ satisfies

$$1 \gtrsim \sum_{k=1}^{r} \left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1\right) \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot \tilde{s}_k, \qquad l = 1, \ldots, r, \tag{7.91}$$

we get (7.89) and (7.87). To see this, note that by (7.61) we have that

$$q_k^1 + q_k^2 + (u-2)\tilde{q}_k \geq q_k, \qquad 1 \leq k \leq r, \tag{7.92}$$

so since $q_k^1 = q_k^2 = \frac{1}{4}q_k$, and by (7.92), (7.90) and the choice of u in (7.62), it follows that

$$
\begin{aligned}
2(8(\lceil\log(\gamma^{-1})+3\lceil\log_2(8KM\sqrt{s})\rceil\rceil)-2)\tilde{q}_k \geq q_k &= \frac{m_k}{N_k - N_{k-1}} \\
&\geq C\frac{\hat{m}_k}{N_k - N_{k-1}}(\log(s\epsilon^{-1})+1)\log\left(KM\sqrt{s}\right) \\
&\geq C\frac{\hat{m}_k}{N_k - N_{k-1}}(\log(s)+1)(\log\left(KM\sqrt{s}\right)+\log(\epsilon^{-1})),
\end{aligned}
$$

for some constant C (recall that we have assumed that $\log(s) \geq 1$). And this gives (by recalling that $\gamma = \epsilon/6$) that $\tilde{q}_k \geq \hat{C}\frac{\hat{m}_k}{N_k - N_{k-1}}(\log(s)+1)$, for some constant $\hat{C}$. Thus, (7.15) implies that for $1 \leq l \leq r$,

$$
\begin{aligned}
1 &\gtrsim (\log(s)+1)\left(\sum_{k=1}^{r}\left(\frac{N_k - N_{k-1}}{m_k(\log(s)+1)} - \frac{1}{\log(s)+1}\right)\cdot \mu_{\mathbf{N},\mathbf{M}}(k,l)\cdot \tilde{s}_k\right) \\
&\gtrsim (\log(s)+1)\left(\sum_{k=1}^{r}\left(\tilde{q}_k^{-1}-1\right)\cdot \mu_{\mathbf{N},\mathbf{M}}(k,l)\cdot \tilde{s}_k\right),
\end{aligned}
$$

and this implies (7.89) and (7.87), given an appropriate choice of the constant C.

Stage 2: (7.14) $\Rightarrow$ (7.88) and (7.86). To show the assertion we must demonstrate that if, for $1 \leq k \leq r$,

$$
1 \gtrsim (\log(s\epsilon^{-1})+1)\cdot\frac{N_k - N_{k-1}}{m_k}\cdot(\sum_{l=1}^{r}\kappa_{\mathbf{N},\mathbf{M}}(k,l))\cdot\log\left(KM\sqrt{s}\right), \tag{7.93}
$$

we obtain (7.88) and (7.86). To see this, note that by arguing as above via the fact that $q_k^1 = q_k^2 = \frac{1}{4}q_k$, and by (7.92), (7.93) and the choice of u in (7.62) we have that

$$
\begin{aligned}
2(8(\lceil\log(\gamma^{-1})+3\lceil\log_2(8KM\sqrt{s})\rceil\rceil)-2)\tilde{q}_k \geq q_k &= \frac{m_k}{N_k - N_{k-1}} \\
&\geq C\cdot(\sum_{l=1}^{r}\kappa_{\mathbf{N},\mathbf{M}}(k,l))\cdot(\log(s\epsilon^{-1})+1)\cdot\log\left(KM\sqrt{s}\right) \\
&\geq C\cdot(\sum_{l=1}^{r}\kappa_{\mathbf{N},\mathbf{M}}(k,l))\cdot(\log(s)+1)\left(\log(\epsilon^{-1})+\log\left(KM\sqrt{s}\right)\right),
\end{aligned}
$$

for some constant C. Thus, we have that for some appropriately chosen constant $\hat{C}$, $\tilde{q}_k \geq \hat{C}\cdot(\log(s)+1)\cdot\sum_{l=1}^{r}\kappa_{\mathbf{N},\mathbf{M}}(k,l)$. So, (7.88) and (7.86) holds given an appropriately chosen C. This yields the last puzzle of the proof, and we are done. □

Proof of Proposition 7.4. The proof is very close to the proof of Proposition 7.3 and we will simply point out the differences. The strategy of the proof is to show the validity of (i) and (ii), and the existence of a $\rho \in \mathrm{ran}(U^*(P_{\Omega_1}\oplus\ldots\oplus P_{\Omega_r}))$ that satisfies (iii)-(v) in Proposition 7.1 with probability exceeding $1-\epsilon$.

Step I: The construction of ρ: The construction is almost identical to the construction in the proof of Proposition 7.3, except that

$$
u = 8\lceil\log(\gamma^{-1})+3v\rceil, \qquad v = \lceil\log_2(8K\tilde{M}\sqrt{s})\rceil, \tag{7.94}
$$

$$
\alpha_1 = \alpha_2 = (2\log_2^{1/2}(4K\tilde{M}\sqrt{s}))^{-1}, \qquad \alpha_i = 1/2, \quad 3 \leq i \leq u,
$$

as well as

$$
\beta_1 = \beta_2 = \frac{1}{4}, \qquad \beta_i = \frac{1}{4}\log_2(4K\tilde{M}\sqrt{s}), \quad 3 \leq i \leq u,
$$

and (7.66) gets changed to

$$
\omega_i = \begin{cases}
\omega_{i-1}\cup\{i\} & \text{if } \|(P_\Delta - P_\Delta U^*(\frac{1}{q_1^i}P_{\Omega_1^i}\oplus\ldots\oplus\frac{1}{q_r^i}P_{\Omega_r^i})UP_\Delta)Z_{i-1}\|_{l^\infty} \leq \alpha_i\|P_{\Delta_k}Z_{i-1}\|_{l^\infty}, \\
 & \text{and } \|P_\Delta^\perp U^*(\frac{1}{q_1^i}P_{\Omega_1^i}\oplus\ldots\oplus\frac{1}{q_r^i}P_{\Omega_r^i})UP_\Delta Z_{i-1}\|_{l^\infty} \leq \beta_i\|Z_{i-1}\|_{l^\infty}, \\
\omega_{i-1} & \text{otherwise,}
\end{cases}
$$

the events B_i, $i = 1, 2$ in (7.67) get replaced by

$$\widetilde{B}_i : \qquad \|P_\Delta^\perp U^*(\frac{1}{q_1^i} P_{\Omega_1^i} \oplus \ldots \oplus \frac{1}{q_r^i} P_{\Omega_r^i}) U P_\Delta Z_{i-1}\|_{l^\infty} \leq \beta_i \|Z_{i-1}\|_{l^\infty}, \qquad i = 1, 2.$$

and the second part of B_3 becomes

$$\max_{i \in \Delta^c} \| \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} P_{\Omega_r} \right) U e_i \| \leq \sqrt{5/4}.$$

Step II: $B_5 \Rightarrow$ (**i**), (**ii**). This step is identical to Step II in the proof of Proposition 7.3.
Step III: $B_5 \Rightarrow$ (**iii**), (**iv**). Equation (7.69) gets changed to

$$\begin{aligned}
\|P_\Delta^\perp \rho\|_{l^\infty} &\leq \sum_{i=1}^{v} \|P_\Delta^\perp U^*(\frac{1}{q_1^{\tau(i)}} P_{\Omega_1^{\tau(i)}} \oplus \ldots \oplus \frac{1}{q_r^{\tau(i)}} P_{\Omega_r^{\tau(i)}}) U P_\Delta Z_{\tau(i-1)}\|_{l^\infty} \\
&\leq \sum_{i=1}^{v} \beta_{\tau(i)} \|Z_{\tau(i-1)}\|_{l^\infty} \leq \sum_{i=1}^{v} \beta_{\tau(i)} \prod_{j=1}^{i-1} \alpha_{\tau(j)} \\
&\leq \frac{1}{4}(1 + \frac{1}{2\log_2^{1/2}(a)} + \frac{\log_2(a)}{2^3 \log_2(a)} + \ldots + \frac{1}{2^{v-1}}) \leq \frac{1}{2}, \qquad a = 4\tilde{M} K \sqrt{s}.
\end{aligned}$$

Step IV: $B_5 \Rightarrow$ (**v**). This step is identical to Step IV in the proof of Proposition 7.3.
Step V: The strong balancing property, (7.18) and (7.19) $\Rightarrow \mathbb{P}(A_1^c \cup A_2^c \cup \widetilde{B}_1^c \cup \widetilde{B}_2^c \cup B_3^c) \leq 5\gamma$. We will start by bounding $\mathbb{P}(\widetilde{B}_1^c)$ and $\mathbb{P}(\widetilde{B}_2^c)$. Note that by Proposition 7.11 (ii) it follows that $\mathbb{P}(\widetilde{B}_1^c) \leq \gamma$ and $\mathbb{P}(\widetilde{B}_2^c) \leq \gamma$ as long as the strong balancing property is satisfied and

$$1 \gtrsim \Lambda \cdot \log\left(\frac{4}{\gamma}(\tilde{\theta} - s)\right), \qquad 1 \gtrsim \Upsilon \cdot \log\left(\frac{4}{\gamma}(\tilde{\theta} - s)\right) \tag{7.95}$$

where $\tilde{\theta} = \tilde{\theta}(\{q_k^i\}_{k=1}^r, 1/8, \{N_k\}_{k=1}^r, s, M)$ for $i = 1, 2$ and where $\tilde{\theta}$ is defined in Proposition 7.11 (ii) and Λ and Υ are defined in (7.73) and (7.74). Note that it is easy to see that we have

$$\left| \left\{ j \in \mathbb{N} : \max_{\substack{\Gamma_1 \subset \{1,\ldots,M\}, \quad |\Gamma_1| = s \\ \Gamma_{2,j} \subset \{N_{j-1}+1,\ldots,N_j\}, \quad j=1,\ldots,r}} \|P_{\Gamma_1} U^*((q_1^i)^{-1} P_{\Gamma_{2,1}} \oplus \ldots \oplus (q_r^i)^{-1} P_{\Gamma_{2,r}}) U e_j\| > \frac{1}{8\sqrt{s}} \right\} \right| \leq \tilde{M},$$

where

$$\tilde{M} = \min\{i \in \mathbb{N} : \max_{j \geq i} \|P_N U P_{\{j\}}\| \leq 1/(K 32 \sqrt{s})\},$$

and this follows from the choice in (7.63) where $q_k^1 = q_k^2 = \frac{1}{4} q_k$ for $1 \leq k \leq r$. Thus, it immediately follows that (7.18) and (7.19) imply (7.95). To bound $\mathbb{P}(B_3^c)$, we first deduce as in Step V of the proof of Proposition 7.3 that

$$\mathbb{P}\left(\|P_\Delta U^*(\frac{1}{q_1} P_{\Omega_1} \oplus \ldots \oplus \frac{1}{q_r} P_{\Omega_r}) U P_\Delta - P_\Delta\| > 1/4, \right) \leq \gamma/2$$

when the strong balancing property and (7.18) holds. For the second part of B_3, we know from the choice of $\tilde{M}$ that

$$\max_{i \geq \tilde{M}} \| \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} P_{\Omega_r} \right) U e_i \| \leq \sqrt{\frac{5}{4}}$$

and we may deduce from Proposition 7.13 that

$$\mathbb{P}\left(\max_{i \in \Delta^c \cap \{1,\ldots,\tilde{M}\}} \| \left(q_1^{-1/2} P_{\Omega_1} \oplus \ldots \oplus q_r^{-1/2} P_{\Omega_r} \right) U e_i \| > \sqrt{5/4} \right) \leq \frac{\gamma}{2},$$

whenever

$$1 \gtrsim \log\left(\frac{2\tilde{M}}{\gamma}\right) \cdot \max_{1 \leq k \leq r} \left\{ \left(\frac{N_k - N_{k-1}}{m_k} - 1 \right) \mu_{\mathbf{N},\mathbf{M}}(k, l) \right\}, \quad l = 1, \ldots, r-1, \infty,$$

which is true whenever (7.18) holds, since by a similar argument to (7.78),

$$\kappa_{\mathbf{N},\mathbf{M}}(k,\infty) + \sum_{j=1}^{r-1} \kappa_{\mathbf{N},\mathbf{M}}(k,j) \geq \mu_{\mathbf{N},\mathbf{M}}(k,l), \qquad l = 1,\ldots,r-1,\infty.$$

Thus, $\mathbb{P}(B_3^c) \leq \gamma$. As for bounding $\mathbb{P}(A_1^c)$ and $\mathbb{P}(A_2^c)$, observe that by the strong balancing property $\tilde{M} \geq M$, thus this is done exactly as in Step V of the proof of Proposition 7.3.

Step VI: The strong balancing property, (7.18) and (7.19) $\Rightarrow$ $\mathbb{P}(B_4^c) \leq \gamma$. To see this, define the random variables $X_1,\ldots X_{u-2}$ as in (7.79). Let π be defined as in Step VI of the proof of Proposition 7.3. Then it suffices to show that (7.18) and (7.19) imply that for $l = 1,\ldots \binom{u-2}{u-v}$ and $j = 0,\ldots,u-v-2$, we have

$$\begin{aligned} &\frac{1}{4} \geq \mathbb{P}(X_{\pi(l)_{u-v-j}} = 1 \,|\, X_{\pi(l)_1} = 1,\ldots,X_{\pi(l)_{u-v-(j+1)}} = 1), \\ &\frac{1}{4} \geq \mathbb{P}(X_{\pi(l)_1} = 1). \end{aligned} \tag{7.96}$$

Claim: The strong balancing property, (7.18) and (7.19) $\Rightarrow$ (7.96). To prove the claim we first observe that $X_j = 0$ when

$$\|(P_\Delta - P_\Delta U^*(\frac{1}{q_1^i}P_{\Omega_1^i} \oplus \ldots \oplus \frac{1}{q_r^i}P_{\Omega_r^i})UP_\Delta)Z_{i-1}\|_{l^\infty} \leq \frac{1}{2}\|Z_{i-1}\|_{l^\infty}$$

$$\|P_\Delta^\perp U^*(\frac{1}{q_1^i}P_{\Omega_1^i} \oplus \ldots \oplus \frac{1}{q_r^i}P_{\Omega_r^i})UP_\Delta Z_{i-1}\|_{l^\infty} \leq \frac{1}{4}\log_2(4K\tilde{M}\sqrt{s})\|Z_{i-1}\|_{l^\infty}, \qquad i = j+2.$$

Thus, by again recalling from (7.63) that $q_k^3 = q_k^4 = \ldots = q_k^u = \tilde{q}_k$, $1 \leq k \leq r$, and by choosing $\tilde{\gamma} = 1/4$ in (7.48) in Proposition 7.12 and $\tilde{\gamma} = 1/4$ in (ii) in Proposition 7.11, we conclude that (7.96) follows when the strong balancing property is satisfied as well as (7.86) and (7.87). and

$$\frac{\log_2(4K\tilde{M}\sqrt{s})}{\log\Big(16(\tilde{M}-s)\Big)} \geq C_2 \cdot \tilde{q}_k^{-1} \cdot \left(\sum_{l=1}^{r-1} \kappa_{\mathbf{N},\mathbf{M}}(k,l) + \kappa_{\mathbf{N},\mathbf{M}}(k,\infty)\right), \quad k = 1,\ldots,r \tag{7.97}$$

$$\frac{\log_2(4K\tilde{M}\sqrt{s})}{\log\Big(16(\tilde{M}-s)\Big)} \geq C_2 \cdot \left(\sum_{l=1}^{r} \left(\tilde{q}_k^{-1} - 1\right) \cdot \mu_{\mathbf{N},\mathbf{M}}(k,l) \cdot \tilde{s}_k\right), \quad l = 1,\ldots,r-1,\infty \tag{7.98}$$

for $K = \max_{1\leq k\leq r}(N_k - N_{k-1})/m_k$. for some constants C_1 and C_2. Thus, to prove the claim we must demonstrate that (7.18) and (7.19) $\Rightarrow$ (7.86), (7.87), (7.97) and (7.98). This is done by repeating Stage 1 and Stage 2 in Step VI of the proof of Proposition 7.3 almost verbatim, except replacing M by $\tilde{M}$. □

7.4 Proof of Theorem 6.2

Throughout this section, we use the notation

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}\mathrm{d}x, \tag{7.99}$$

to denote the Fourier transform of a function $f \in L^1(\mathbb{R})$.

7.4.1 Setup

We first introduce the wavelet sparsity and Fourier sampling bases that we consider, and in particular, their orderings. Consider an orthonormal basis of compactly supported wavelets with an MRA [22, 23]. For simplicity, suppose that $\mathrm{supp}(\Psi) = \mathrm{supp}(\Phi) = [0,a]$ for some $a \geq 1$, where Ψ and Φ are the mother wavelet and scaling function respectively. For later use, we recall the following three properties of any such wavelet basis:

1. There exist $\alpha \geq 1$, C_Ψ and $C_\Phi > 0$, such that

$$\left|\hat{\Phi}(\xi)\right| \leq \frac{C_\Phi}{(1+|\xi|)^\alpha}, \quad \left|\hat{\Psi}(\xi)\right| \leq \frac{C_\Psi}{(1+|\xi|)^\alpha}. \tag{7.100}$$

   See [23, Eqn. (7.1.4)]. We will denote $\max\{C_\Psi, C_\Phi\}$ by $C_{\Phi,\Psi}$.

2. Ψ has $v \geq 1$ vanishing moments and $\hat{\Psi}(z) = (-iz)^v \theta_\Psi(z)$ for some bounded function θ_Ψ (see [56, p.208 & p.284].

3. $\|\hat{\Phi}\|_{L^\infty}, \|\hat{\Psi}\|_{L^\infty} \leq 1$.

We now wish to construct a wavelet basis for the compact interval $[0, a]$. The most standard approach is to consider the following collection of functions

$$\Omega_a = \{\Phi_k, \Psi_{j,k} : \mathrm{supp}(\Phi_k)^o \cap [0,a] \neq \emptyset,\ \mathrm{supp}(\Psi_{j,k})^o \cap [0,a] \neq \emptyset, j \in \mathbb{Z}_+, k \in \mathbb{Z}, \},$$

where

$$\Phi_k = \Phi(\cdot - k), \qquad \Psi_{j,k} = 2^{\frac{j}{2}} \Psi(2^j \cdot - k).$$

(the notation K^o denotes the interior of a set $K \subseteq \mathbb{R}$). This now gives

$$\left\{f \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(f) \subseteq [0,a]\right\} \subseteq \overline{\mathrm{span}\{\varphi : \varphi \in \Omega_a\}} \subseteq \left\{f \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(f) \subseteq [-T_1, T_2]\right\},$$

where $T_1, T_2 > 0$ are such that $[-T_1, T_2]$ contains the support of all functions in Ω_a. Note that the inclusions may be proper (but not always, as is the case with the Haar wavelet). It is easy to see that

$$\begin{aligned} \Psi_{j,k} \notin \Omega_a &\Longleftrightarrow \frac{a+k}{2^j} \leq 0, \quad a \leq \frac{k}{2^j}, \\ \Phi_k \notin \Omega_a &\Longleftrightarrow a + k \leq 0, \quad a \leq k, \end{aligned}$$

and therefore

$$\Omega_a = \{\Phi_k : |k| = 0, \ldots, \lceil a \rceil - 1\} \cup \{\Psi_{j,k} : j \in \mathbb{Z}_+, k \in \mathbb{Z}, -\lceil a \rceil < k < 2^j \lceil a \rceil\}.$$

We order Ω_a in increasing order of wavelet resolution as follows:

$$\begin{aligned} &\{\Phi_{-\lceil a \rceil+1}, \ldots, \Phi_{-1}, \Phi_0, \Phi_1, \ldots, \Phi_{\lceil a \rceil - 1}, \\ &\Psi_{0,-\lceil a \rceil+1}, \ldots, \Psi_{0,-1}, \Psi_{0,0}, \Psi_{0,1}, \ldots, \Psi_{0,\lceil a \rceil-1}, \Psi_{1,-\lceil a \rceil+1}, \ldots\}. \end{aligned} \tag{7.101}$$

By the definition of Ω_a, we let $T_1 = \lceil a \rceil - 1$ and $T_2 = 2\lceil a \rceil - 1$. Finally, for $R \in \mathbb{N}$ let $\Omega_{R,a}$ contain all wavelets in Ω_a with resolution less than R, so that

$$\Omega_{R,a} = \{\varphi \in \Omega_a : \varphi = \Psi_{j,k},\ j < R, \text{ or } \varphi = \Phi_k\}. \tag{7.102}$$

We also denote the size of $\Omega_{R,a}$ by W_R. It is easy to verify that

$$W_R = 2^R \lceil a \rceil + (R+1)(\lceil a \rceil - 1). \tag{7.103}$$

Having constructed an orthonormal wavelet system form $[0, a]$ we now introduce the appropriate Fourier sampling basis. We must sample at at least the Nyquist rate. Hence we let $\omega \leq 1/(T_1 + T_2)$ be the *sampling density* (note that $1/(T_1 + T_2)$ is the Nyquist criterion for functions supported on $[-T_1, T_2]$). For simplicity, we assume throughout that

$$\omega \in (0, 1/(T_1 + T_2)), \quad \omega^{-1} \in \mathbb{N}, \tag{7.104}$$

and remark that this assumption is an artefact of our proofs and is not necessary in practice. The Fourier sampling vectors are now defined as follows.

$$\psi_j(x) = \sqrt{\omega} e^{2\pi i j \omega x} \chi_{[-T_1/(\omega(T_1+T_2)), T_2/(\omega(T_1+T_2))]}(x), \qquad j \in \mathbb{Z}. \tag{7.105}$$

This gives an orthonormal sampling basis for the space $\{f \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(f) \subseteq [-T_1, T_2]\}$. Since Ω_a is an orthonormal system in for this space, it follows that the infinite matrix $U = \{\langle \varphi_i, \tilde{\psi}_j \rangle\}_{i,j \in \mathbb{N}}$ is an isometry, where $\{\varphi_j\}_{j \in \mathbb{N}}$ represents the wavelets ordered according to (7.101) and $\{\tilde{\psi}_j\}_{j \in \mathbb{N}}$ is the standard ordering of the Fourier basis (7.105) over $\mathbb{N}$ ($\tilde{\psi}_1 = \psi_0$, $\tilde{\psi}_{2n} = \psi_n$ and $\tilde{\psi}_{2n+1} = \psi_{-n}$).

7.4.2 Some preliminary estimates

Throughout this section, we assume the setup and notation introduced above.

Theorem 7.15. *Let U be the matrix of the Fourier/wavelets pair introduced previously. Then*

(i) We have $\mu(U) \geq \omega|\hat{\Phi}(0)|^2 > 0$, where Φ is the corresponding scaling function.

(ii) We have

$$\mu(P_N^{\perp}U) \leq \frac{C_{\Phi,\Psi}^2}{\pi N(2\alpha-1)(1+1/(2\alpha-1))^{2\alpha}}, \qquad \mu(UP_N^{\perp}) \leq 4\frac{\omega\lceil a\rceil}{N},$$

and consequently $\mu(P_N^{\perp}U), \mu(UP_N^{\perp}) = \mathcal{O}\left(N^{-1}\right)$ as $N \to \infty$.

(iii) If the wavelet and scaling function satisfy decay estimate (7.100) with $\alpha > 1/2$, then, for R such that $\omega^{-1}2^R \leq N$ and $M = |\Omega_{R,a}|$ (recalling the definition of $\Omega_{R,a}$ from (7.102)),

$$\mu(P_N^{\perp}UP_M) \leq \frac{C_{\Phi,\Psi}^2}{\pi^{2\alpha}\omega^{2\alpha-1}}(2^{R-1}N^{-1})^{2\alpha-1}N^{-1}.$$

(iv) If the wavelet has $v \geq 1$ vanishing moments, $\omega^{-1}2^R \geq N$ and $M = |\Omega_{R,a}|$ with $R \geq 1$, then

$$\mu(P_NUP_M^{\perp}) \leq \frac{\omega}{2^R} \cdot \left(\frac{\pi\omega N}{2^R}\right)^{2v} \cdot \|\theta_\Psi\|_{L^\infty}^2,$$

where θ_Ψ is the function such that $\hat{\Psi}(z) = (-iz)^v\theta_\Psi(z)$ (see above).

Proof. Note that $\mu(U) \geq |\langle\Phi,\psi_0\rangle|^2 = \omega\left|\hat{\Phi}(0)\right|^2$, moreover, it is known that $\hat{\Phi}(0) \neq 0$ [44, Thm. 1.7]. Thus, (i) follows.

To show (ii), first note that for $R \in \mathbb{N}$, $j,k \in \mathbb{Z}$,

$$\langle\Psi_{R,j},\psi_k\rangle = \sqrt{\frac{\omega}{2^R}}\hat{\Psi}\left(\frac{-2\pi k\omega}{2^R}\right)e^{2\pi i\omega kj/2^R}, \quad \langle\Phi_j,\psi_k\rangle = \sqrt{\omega}\hat{\Phi}\left(-2\pi k\omega\right)e^{2\pi i\omega kj}.$$

Thus, the decay estimate in (7.100) yields

$$\begin{aligned}
\mu(P_N^{\perp}U) &\leq \max_{|k|\geq\frac{N}{2}}\max_{\varphi\in\Omega_a}|\langle\varphi,\psi_k\rangle|^2 \\
&= \max\left\{\max_{|k|\geq\frac{N}{2}}\max_{R\in\mathbb{Z}_+}\frac{\omega}{2^R}\left|\hat{\Psi}\left(\frac{-2\pi\omega k}{2^R}\right)\right|^2, \omega\max_{|k|\geq\frac{N}{2}}\left|\hat{\Phi}\left(-2\pi\omega k\right)\right|^2\right\} \\
&\leq \max_{|k|\geq\frac{N}{2}}\max_{R\in\mathbb{Z}_+}\frac{\omega}{2^R}\frac{C_{\Phi,\Psi}^2}{(1+|2\pi\omega k2^{-R}|)^{2\alpha}} \leq \max_{R\in\mathbb{Z}_+}\frac{\omega}{2^R}\frac{C_{\Phi,\Psi}^2}{(1+|\pi\omega N2^{-R}|)^{2\alpha}}.
\end{aligned}$$

The function $f(x) = x^{-1}(1+\pi\omega N/x)^{-2\alpha}$ on $[1,\infty)$ satisfies $f'(\pi\omega N(2\alpha-1)) = 0$. Hence

$$\mu(P_N^{\perp}U) \leq \frac{C_{\Phi,\Psi}^2}{\pi N(2\alpha-1)(1+1/(2\alpha-1))^{2\alpha}},$$

which gives the first part of (ii). For the second part, we first recall the definition of W_R for $R \in \mathbb{N}$ from (7.103). Then, given any $N \in \mathbb{N}$ such that $N \geq W_1$, let R be such that $W_R \leq N < W_{R+1}$. Then, for each $n \geq N$, there exists some $j \geq R$ and $l \in \mathbb{Z}$ such that the nth element via the ordering (7.101) is $\varphi_n = \Psi_{j,l}$. Hence

$$\begin{aligned}
\mu(UP_N^{\perp}) &= \max_{n\geq N}\max_{k\in\mathbb{Z}}|\langle\varphi_n,\psi_k\rangle|^2 = \max_{j\geq R}\max_{k\in\mathbb{Z}}\frac{\omega}{2^j}\left|\hat{\Psi}\left(\frac{-2\pi\omega k}{2^j}\right)\right|^2 \\
&\leq \|\hat{\Psi}\|_{L^\infty}^2\frac{\omega}{2^R} \leq 4\|\hat{\Psi}\|_{L^\infty}^2\frac{\omega\lceil a\rceil}{N},
\end{aligned}$$

where the last line follows because $N < W_{R+1} = 2^{R+1}\lceil a\rceil + (R+2)(\lceil a\rceil - 1)$ implies that

$$2^{-R} < \frac{1}{N}\left(2\lceil a\rceil + (R+2)(\lceil a\rceil - 1)2^{-R}\right) \leq \frac{4\lceil a\rceil}{N}.$$

This concludes the proof of (ii).

To show (iii), observe that the decay estimate in (7.100) yields

$$\begin{aligned}
\mu(P_N^{\perp} U P_{W_R}) &\leq \max_{|k|\geq \frac{N}{2}} \max_{\varphi\in\Omega_{R,a}} |\langle \varphi, \psi_k\rangle|^2 \\
&= \max\left\{ \max_{|k|\geq\frac{N}{2}} \max_{j<R} \frac{\omega}{2^j} \left|\hat{\Psi}\left(\frac{-2\pi\omega k}{2^j}\right)\right|^2, \max_{|k|\geq\frac{N}{2}} \left|\hat{\Phi}\left(-2\pi\omega k\right)\right|^2 \right\} \\
&\leq \max_{|k|\geq\frac{N}{2}} \max_{j<R} \frac{\omega}{2^j} \frac{C_{\Phi,\Psi}^2}{(1+|2\pi\omega k 2^{-j}|)^{2\alpha}} \leq \max_{k\geq\frac{N}{2}} \max_{j<R} \frac{C_{\Phi,\Psi}^2}{\pi^{2\alpha}\omega^{2\alpha-1}} \frac{2^{j(2\alpha-1)}}{(2k)^{2\alpha}} \\
&= \frac{C_{\Phi,\Psi}^2}{\pi^{2\alpha}\omega^{2\alpha-1}} (2^{R-1}N^{-1})^{2\alpha-1} N^{-1}.
\end{aligned}$$

To show (iv), first note that because $R \geq 1$, for all $n > W_R$, $\varphi_n = \Psi_{j,k}$ for some $j \geq 0$ and $k \in \mathbb{Z}$. Then, recalling the properties of Daubechies wavelets with v vanishing moments,

$$\begin{aligned}
\mu(P_N U P_{W_R}^{\perp}) &= \max_{n>W_R} \max_{|k|\leq\frac{N}{2}} |\langle \varphi_n, \psi_k\rangle|^2 = \max_{j\geq R} \max_{|k|\leq\frac{N}{2}} \frac{\omega}{2^j} \left|\hat{\Psi}\left(\frac{-2\pi\omega k}{2^j}\right)\right|^2 \\
&\leq \frac{\omega}{2^R} \cdot \left(\frac{\pi\omega N}{2^R}\right)^{2v} \cdot \|\theta_\Psi\|_{L^\infty}^2,
\end{aligned}$$

as required. □

Corollary 7.16. *Let* $\mathbf{N}$ *and* $\mathbf{M}$ *be as in Theorem 6.2 and recall the definition of* $\mu_{\mathbf{N},\mathbf{M}}(k,j)$ *in (4.2). Then,*

$$\mu_{\mathbf{N},\mathbf{M}}(k,j) \leq B_{\Phi,\Psi} \cdot \begin{cases} \frac{\sqrt{\omega}}{\sqrt{N_{k-1}2^{R_{j-1}}}} \cdot \left(\frac{\omega N_k}{2^{R_{j-1}}}\right)^v & j \geq k+1 \\ \frac{1}{N_{k-1}} \left(\frac{2^{R_{j-1}}}{\omega N_{k-1}}\right)^{\alpha-1/2} & j \leq k-1 \\ \frac{1}{N_{k-1}} & j = k \end{cases} \tag{7.106}$$

$$\mu_{\mathbf{N},\mathbf{M}}(k,\infty) \leq B_{\Phi,\Psi} \cdot \begin{cases} \frac{\sqrt{\omega}}{\sqrt{N_{k-1}2^{R_{r-1}}}} \cdot \left(\frac{\omega N_k}{2^{R_{r-1}}}\right)^v & k \leq r-1 \\ \frac{1}{N_{r-1}} & k = r. \end{cases} \tag{7.107}$$

where $B_{\Phi,\Psi}$ *is a constant which depends only on* Φ *and* Ψ*.*

Proof. Throughout this proof, $B_{\Phi,\Psi}$ is a constant which depends only on Φ and Ψ, although its value may change from instance to instance. First note that, for $k, j \in \{1, \ldots, r\}$,

$$\mu_{\mathbf{N},\mathbf{M}}(k,j) = \sqrt{\mu(P_{N_k}^{N_{k-1}} U P_{M_j}^{M_{j-1}}) \cdot \mu(P_{N_k}^{N_{k-1}} U)} \leq B_{\Phi,\Psi} N_{k-1}^{-1/2} \sqrt{\mu(P_{N_k}^{N_{k-1}} U P_{M_j}^{M_{j-1}})}, \tag{7.108}$$

because by (ii) of Theorem 7.15, $\mu(P_{N_{k-1}}^{\perp} U) \leq B_{\Phi,\Psi} N_{k-1}^{-1}$. Thus,

$$\mu_{\mathbf{N},\mathbf{M}}(k,k) \leq \mu(P_{N_{k-1}}^{\perp} U) \leq B_{\Phi,\Psi} \frac{1}{N_{k-1}},$$

yielding the last part of (7.106). As for the middle part of (7.106), note that if $j \leq k-1$, then by applying (iii) of Theorem 7.15, we obtain

$$\sqrt{\mu(P_{N_k}^{N_{k-1}} U P_{M_j}^{M_{j-1}})} \leq \sqrt{\mu(P_{N_{k-1}}^{\perp} U P_{M_j})} \leq B_{\Phi,\Psi} \cdot \frac{1}{\sqrt{N_{k-1}}} \left(\frac{2^{R_{j-1}}}{\omega N_{k-1}}\right)^{\alpha-1/2},$$

and thus, in combination with (7.108), we obtain the $j \leq k-1$ part of (7.106). If $j \geq k+1$, then by applying (iv) of Theorem 7.15, we obtain

$$\sqrt{\mu(P_{N_k}^{N_{k-1}} U P_{M_j}^{M_{j-1}})} \leq \sqrt{\mu(P_{N_k} U P_{M_{j-1}}^{\perp})} \leq B_{\Phi,\Psi} \cdot \frac{\sqrt{\omega}}{\sqrt{2^{R_{j-1}}}} \cdot \left(\frac{\omega N_k}{2^{R_{j-1}}}\right)^v,$$

and in combination with (7.108), we obtain the $j \geq k+1$ part of (7.106). Finally, recall that

$$\mu_{\mathbf{N},\mathbf{M}}(k,\infty) = \sqrt{\mu(P_{N_k}^{N_{k-1}} U P_{M_{r-1}}^{\perp}) \cdot \mu(P_{N_{k-1}}^{\perp} U)}$$

and similarly to the above, (7.107) is a direct consequence of parts (ii) and (iv) of Theorem 7.15. □

Lemma 7.17 ([60]). *The following holds:*

(i) If there exists $C > 0$ and $\alpha \geq 1$ such that

$$\left|\hat{\Phi}(\xi)\right| \leq \frac{C}{(1+|\xi|)^\alpha}, \quad \xi \in \mathbb{R}, \tag{7.109}$$

then N, K satisfy the strong balancing property with respect to U, M and s whenever $N \gtrsim M^{1+1/(2\alpha-1)} \cdot (\log_2(4MK\sqrt{s}))^{1/(2\alpha-1)}$.

(ii) If, for some $C > 0$ and $\alpha \geq 1.5$,

$$\left|\hat{\Phi}^{(k)}(\xi)\right| \leq \frac{C}{(1+|\xi|)^\alpha}, \quad \left|\hat{\Psi}^{(k)}(\xi)\right| \leq \frac{C}{(1+|\xi|)^\alpha}, \quad \xi \in \mathbb{R}, \qquad k = 0,1,2, \tag{7.110}$$

then N, K satisfy the strong balancing property with respect to U, M and s whenever $N \gtrsim M \cdot (\log_2(4MK\sqrt{s}))^{1/(2\alpha-1)}$.

The following lemma informs us of the range of Fourier samples required for accurate reconstruction of wavelet coefficients.

Lemma 7.18 ([6, 60]). *Let φ_k denote the kth wavelet the ordering in (7.101). Let $M \leq W_R$ be such that $\{\varphi_j : j \leq M\} \subset \Omega_{R,a}$. Then*

$$\left\|P_N^{\perp} U P_M\right\| \leq \min\left\{\gamma_\Phi, \sqrt{\frac{2}{2\alpha-1}} \cdot C_\Phi \cdot \frac{1}{(2\pi)^\alpha} \cdot \frac{1}{L^{\alpha-1/2}}\right\} < 1$$

where N is such that $N \geq L\omega^{-1}2^R$ for some $L \in \mathbb{N}$ and $\gamma_\Phi = \sqrt{1 - \inf_{|\xi|\leq\pi}\left|\hat{\Phi}(\xi)\right|^2} < 1$.

Lemma 7.19. *Let φ_k denote the kth wavelet the ordering in (7.101). Let $R_1, R_2 \in \mathbb{N} \setminus \{0\}$ with $R_2 > R_1$, and let M_1, M_2 be such that*

$$\{\varphi_j : M_2 \geq j > M_1\} \subset \Omega_{R_2,a} \setminus \Omega_{R_1,a}.$$

Then for any $\gamma \in (0,1)$

$$\left\|P_N U P_{M_2}^{M_1}\right\| \leq \frac{\pi^2}{4}\|\theta_\Psi\|_{L^\infty} \cdot (2\pi\gamma)^v \cdot \sqrt{\frac{1-2^{2v(R_1-R_2)}}{1-2^{-2v}}}$$

whenever N is such that $N \leq \gamma\omega^{-1}2^{R_1}$.

Proof. Let $\eta \in l^2(\mathbb{N})$ be such that $\|\eta\| = 1$ and let $\Delta_l \subset \mathbb{N}$ be such that $\{\varphi_j : j \in \Delta_l\} = \{j \in \mathbb{Z} : \Psi_{l,j} \in \Omega_a\}$. First observe that given $l \in \mathbb{N}$, $k, j \in \mathbb{Z}$, $\langle \psi_k, \Psi_{l,j}\rangle = \sqrt{\frac{\omega}{2^l}}\hat{\Psi}\left(-\frac{2\pi\omega k}{2^l}\right)e^{2\pi i\omega jk}$. So,

$$\|P_N U P_{M_2}^{M_1}\eta\|^2 \leq \sum_{|k|\leq N/2}\left|\langle \psi_k, \sum_{l=R_1}^{R_2-1}\sum_{j\in\Delta_l}\eta_j\varphi_j\rangle\right|^2$$

$$\sum_{|k|\leq N/2}\left|\sum_{l=R_1}^{R_2-1}\frac{\sqrt{\omega}}{\sqrt{2^l}}\sum_{j\in\Delta_l}\eta_j\hat{\Psi}\left(-\frac{2\pi\omega k}{2^l}\right)e^{2\pi i\omega jk/2^l}\right|^2,$$

yielding that

$$\|P_N U P_{M_2}^{M_1}\eta\|^2 = \sum_{|k|\le N/2}\left|\sum_{l=R_1}^{R_2-1}\frac{\sqrt{\omega}}{\sqrt{2^l}}\hat{\Psi}\left(-\frac{2\pi\omega k}{2^l}\right)f^{[l]}\left(\frac{\omega k}{2^l}\right)\right|^2$$
$$\le \sum_{l=R_1}^{R_2-1}\max_{|k|\le N/2}\left|\hat{\Psi}\left(-\frac{2\pi\omega k}{2^l}\right)\right|^2\cdot\sum_{l=R_1}^{R_2-1}\sum_{|k|\le N/2}\frac{\omega}{2^l}\left|f^{[l]}\left(\frac{\omega k}{2^l}\right)\right|^2,$$

where $f^{[l]} = \sum_{j\in\Delta_l}\eta_j e^{2\pi izj}$. If we let $H = \chi_{[0,1)}$, then it is known that $\inf_{|x|\le\pi}\left|\hat{H}(x)\right| \ge 2/\pi$, and since $N \le 2^{R_1}/\omega$, for each $l \ge R_1$, we have that

$$\sum_{|k|\le N/2}\frac{\omega}{2^l}\left|f^{[l]}\left(\frac{\omega k}{2^l}\right)\right|^2 \le \left(\inf_{|x|\le\pi}\left|\hat{H}(x)\right|^2\right)^{-1}\sum_{|k|\le N/2}\left|\langle\sum_{j\in\Delta_l}\eta_j H_{l,j},\psi_k\rangle\right|^2$$
$$\le \frac{\pi^2}{4}\left\|\sum_{j\in\Delta_l}\eta_j H_{l,j}\right\|^2 \le \frac{\pi^2}{4}\|P_{\Delta_l}\eta\|^2$$

which yields

$$\sum_{l=R_1}^{R_2-1}\sum_{|k|\le N/2}\frac{\omega}{2^l}\left|f^{[l]}\left(\frac{\omega k}{2^l}\right)\right|^2 \le \frac{\pi^2}{4}\sum_{l=R_1}^{R_2-1}\|P_{\Delta_l}\eta\|^2 \le \frac{\pi^2}{4}\|\eta\|^2 \le \frac{\pi^2}{4}.$$

Also, since Ψ has v vanishing moments, we have that $\hat{\Psi}(z) = (-iz)^v\theta(z)$ for some bounded function θ. Thus, since $N \le \gamma\cdot 2^{R_1}/\omega$, we have

$$\sum_{l=R_1}^{R_2-1}\max_{|k|\le N/2}\left|\hat{\Psi}\left(\frac{2\pi\omega k}{2^l}\right)\right|^2 \le \frac{\pi^2}{4}\|\theta_\Psi\|_{L^\infty}^2\sum_{l=R_1}^{R_2-1}\left(2\pi\gamma 2^{R_1-l}\right)^{2v}$$
$$\le \frac{\pi^2}{4}(2\pi\gamma)^{2v}\|\theta_\Psi\|_{L^\infty}^2\frac{1-2^{2v(R_1-R_2)}}{1-2^{-2v}}.$$

Thus,

$$\|P_N U P_{M_2}^{M_1}\eta\|^2 \le \frac{\pi^2}{4}\|\theta_\Psi\|_{L^\infty}^2\cdot(2\pi\gamma)^{2v}\frac{1-2^{2v(R_1-R_2)}}{1-2^{-2v}}.$$

□

7.4.3 The proof

Proof of Theorem 6.2. In this proof, we will let $B_{\Phi,\Psi}$ be some constant which depends only on Φ and Ψ, although its value may change from instance to instance. The assertions of the theorem will follow if we can show that the conditions in Theorem 5.3 are satisfied. We will begin with condition (i). Note that by Lemma 7.17 (i) we have that for $\alpha \ge 1$, if $N \gtrsim M^{1+1/(2\alpha-1)}\cdot(\log_2(4MK\sqrt{s}))^{1/(2\alpha-1)}$ then N, K satisfy the strong balancing property with respect to U, M, s. Also, the same is true if (6.1) holds and $N \gtrsim M\cdot(\log_2(4KM\sqrt{s}))^{1/(2\alpha-1)}$. In particular, condition (i) implies condition (i) of Theorem 5.3.

To show that (ii) in Theorem 5.3 is satisfied we need to demonstrate that

$$1 \gtrsim \frac{N_k - N_{k-1}}{m_k}\cdot\log(\epsilon^{-1})\cdot\left(\sum_{l=1}^{r}\mu_{\mathbf{N},\mathbf{M}}(k,l)\cdot s_l\right)\cdot\log\left(K\tilde{M}\sqrt{s}\right), \tag{7.111}$$

(with $\mu_{\mathbf{N},\mathbf{M}}(k,r)$ replaced by $\mu_{\mathbf{N},\mathbf{M}}(k,\infty)$) and

$$\begin{aligned} m_k &\gtrsim \hat{m}_k\cdot\log(\epsilon^{-1})\cdot\log\left(K\tilde{M}\sqrt{s}\right), \\ 1 &\gtrsim \sum_{k=1}^{r}\left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1\right)\cdot\mu_{\mathbf{N},\mathbf{M}}(k,l)\cdot\tilde{s}_k, \qquad \forall\, l = 1,\ldots,r. \end{aligned} \tag{7.112}$$

We will first consider (7.111). By applying the bounds on the local coherences derived in Corollary 7.16, we have that (7.111) is implied by

$$\frac{m_k}{(N_k - N_{k-1})} \gtrsim B_{\Phi,\Psi} \cdot \left(\sum_{j=1}^{k-1} \frac{s_j}{N_{k-1}} \left(\frac{2^{R_{j-1}}}{\omega N_{k-1}} \right)^{\alpha - 1/2} + \frac{s_k}{N_{k-1}} \right.$$
$$\left. + \sum_{j=k+1}^{r} \frac{s_j \cdot \sqrt{\omega}}{\sqrt{N_{k-1} 2^{R_{j-1}}}} \cdot \left(\frac{\omega N_k}{2^{R_{j-1}}} \right)^{v} \right) \log(\epsilon^{-1}) \cdot \log\left(K \tilde{M} \sqrt{s} \right).$$

To obtain a bound on the value of $\tilde{M}$, observe that by Lemma 7.19, $\left\| P_N U P_{\{j\}} \right\| \leq 1/(32K\sqrt{s})$ whenever $j = 2^J$ such that $2^J \geq (32K\sqrt{s})^{1/v} \cdot N \cdot \omega$. Thus, $\tilde{M} \leq \lceil (32K\sqrt{s})^{1/v} \cdot N \cdot \omega \rceil$, and by recalling that $N_k = 2^{R_k} \omega^{-1}$, we have that (7.111) is implied by

$$\frac{m_k \cdot N_{k-1}}{N_k - N_{k-1}} \gtrsim B_{\Phi,\Psi} \cdot (\log(\epsilon^{-1}) + 1) \cdot \log\left((K\sqrt{s})^{1+1/v} N \right) \cdot \left(\sum_{j=1}^{k-1} s_j \cdot \left(2^{\alpha - 1/2} \right)^{-(R_{k-1} - R_{j-1})} + s_k \right.$$
$$\left. + s_{k+1} \cdot 2^{-(R_k - R_{k-1})/2} + \sum_{j=k+2}^{r} s_j \cdot 2^{-(R_{j-1} - R_{k-1})/2} \left(2^{v - 1/2} \right)^{-(R_{j-1} - R_k)} \right).$$

and we have derived condition (6.2).

As for condition (7.112), we will first derive upper bounds for the $\tilde{s}_k$ values: By Lemma 7.18,

$$\| P_{N_{k-1}}^{\perp} U P_{M_l} \| < \min\left\{ 1, \sqrt{\frac{2}{2\alpha - 1}} \cdot \frac{C_\Phi}{(2\pi)^\alpha} \cdot \left(\frac{2^{R_l}}{2^{R_{k-1}}} \right)^{\alpha - 1/2} \right\}, \qquad l \leq k - 1.$$

Also, by Lemma 7.19,

$$\| P_{N_k} U P_{M_l}^{M_{l-1}} \eta \| < \min\left\{ 1, \, (2\pi)^v \cdot \| \theta_\Psi \|_{L^\infty} \cdot \left(\frac{2^{R_k}}{2^{R_{l-1}}} \right)^{v} \right\}, \qquad l \geq k + 1.$$

Consequently, for $k = 1, \ldots, r$

$$\sqrt{\tilde{s}_k} \leq \sqrt{S_k} = \max_{\eta \in \Theta} \| P_{N_k}^{N_{k-1}} U \eta \| \leq \sum_{l=1}^{r} \| P_{N_k}^{N_{k-1}} U P_{M_l}^{M_{l-1}} \| \sqrt{s_l}$$
$$\leq B_{\Phi,\Psi} \cdot \left(\sum_{l=1}^{k-2} \sqrt{s_l} \cdot \left(\frac{2^{R_l}}{2^{R_{k-1}}} \right)^{\alpha - 1/2} + \sqrt{s_{k-1}} + \sqrt{s_k} + \sqrt{s_{k+1}} + \sum_{l=k+2}^{r} \sqrt{s_l} \cdot \left(\frac{2^{R_k}}{2^{R_{l-1}}} \right)^{v} \right).$$

Thus, for $A_\alpha = 2^{\alpha - 1/2}$ and $A_v = 2^v$

$$\tilde{s}_k \leq B_{\Phi,\Psi} \cdot \left(\sqrt{s_{k-1}} + \sqrt{s_k} + \sqrt{s_{k+1}} + \sum_{l=1}^{k-2} \sqrt{s_l} \cdot A_\alpha^{-(R_{k-1} - R_l)} + \sum_{l=k+2}^{r} \sqrt{s_l} \cdot A_v^{-(R_{l-1} - R_k)} \right)^2$$
$$\leq B_{\Phi,\Psi} \cdot \left(3 + \sum_{l=1}^{k-2} A_\alpha^{-(R_{k-1} - R_l)} + \sum_{l=k+2}^{r} A_v^{-(R_{l-1} - R_k)} \right)$$
$$\cdot \left(S_k + \sum_{l=1}^{k-2} s_l \cdot A_\alpha^{-(R_{k-1} - R_l)} + \sum_{l=k+2}^{r} s_l \cdot A_v^{-(R_{l-1} - R_k)} \right)$$
$$\leq B_{\Phi,\Psi} \cdot \left(S_k + \sum_{l=1}^{k-2} s_l \cdot A_\alpha^{-(R_{k-1} - R_l)} + \sum_{l=k+2}^{r} s_l \cdot A_v^{-(R_{l-1} - R_k)} \right)$$

where the second inequality is due to the Cauchy-Schwarz inequality and $S_k = s_k + s_{k-1} + s_{k+1}$. Finally, we will show that condition (6.2) implies (4.5): By our coherence estimates in (7.106) and (7.107), we see

that (4.5) holds if $m_k \gtrsim \hat{m}_k \cdot (\log(\epsilon^{-1}) + 1) \cdot \log\left((K\sqrt{s})^{1+1/v} N\right)$ and for each $l = 1, \ldots, r$,

$$\begin{aligned} 1 \gtrsim & B_{\Phi,\Psi} \cdot \left(\sum_{k=1}^{l-1} \left(\frac{N_l - N_{l-1}}{\hat{m}_l} - 1 \right) \cdot \tilde{s}_k \cdot \sqrt{\frac{\omega}{N_{k-1} 2^{R_{l-1}}}} \cdot \left(\frac{\omega N_k}{2^{R_{l-1}}} \right)^v \right. \\ & + \left(\frac{N_l - N_{l-1}}{\hat{m}_l} - 1 \right) \cdot \tilde{s}_l \cdot \frac{1}{N_{l-1}} \\ & \left. + \sum_{k=l+1}^{r} \left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1 \right) \cdot \tilde{s}_k \cdot \frac{1}{N_{k-1}} \left(\frac{2^{R_{l-1}}}{\omega N_{k-1}} \right)^{\alpha - 1/2} \right). \end{aligned} \tag{7.113}$$

Recalling that $N_k = \omega^{-1} 2^{R_k}$, (7.113) becomes

$$\begin{aligned} 1 \gtrsim & B_{\Phi,\Psi} \cdot \left(\sum_{k=1}^{l-1} \left(\frac{N_l - N_{l-1}}{\hat{m}_l} - 1 \right) \cdot \frac{\tilde{s}_k}{N_{k-1}} \cdot (2^v)^{-(R_{l-1} - R_k)} \right. \\ & \left. + \left(\frac{N_l - N_{l-1}}{\hat{m}_l} - 1 \right) \cdot \frac{\tilde{s}_l}{N_{l-1}} + \sum_{k=l+1}^{r} \left(\frac{N_k - N_{k-1}}{\hat{m}_k} - 1 \right) \cdot \frac{\tilde{s}_k}{N_{k-1}} \cdot \left(2^{\alpha - 1/2} \right)^{-(R_{k-1} - R_{l-1})} \right). \end{aligned}$$

Observe that

$$1 + \sum_{k=1}^{l-1} (2^v)^{-(R_{l-1} - R_k)} + \sum_{k=l+1}^{r} \left(2^{\alpha - 1/2} \right)^{-(R_{k-1} - R_{l-1})} \leq B_{\Phi,\Psi}.$$

Thus, (4.5) holds provided that for each $k = 1, \ldots, r$,

$$\hat{m}_k \geq B_{\Phi,\Psi} \cdot \frac{N_k - N_{k-1}}{N_{k-1}} \cdot \tilde{s}_k,$$

and combining with our estimates of $\tilde{s}_k$, we may deduce that (6.2) implies (4.5). □

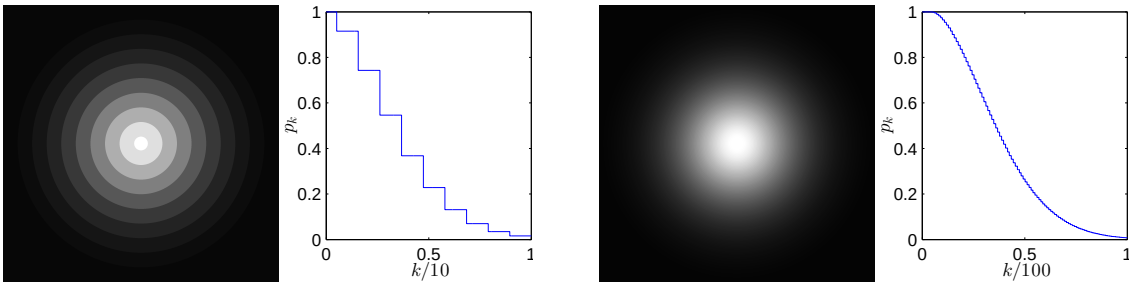
8 Numerical examples

We present here examples of the new theoretical concepts introduced in this paper, discuss implications for traditional compressed sensing concepts and showcase a series of real-world phenomena and applications that can benefit from the new theory. This section attempts to provide an insight into real-world conditions and phenomena, which often exhibit asymptotic sparsity and asymptotic incoherence, into the markedly improved results one can obtain by exploiting this structure, and into how to do it. For those purposes, the section investigates a variety of different sampling and sparsifying bases, sampling techniques and applications. We used the SPGL1 solver [73, 74] for all the compressed sensing experiments. This section concentrates on 2D signals, but the new theory is completely general.

8.1 Multi-level subsampling scheme

As already explained, the optimum sampling strategy is highly dependent on the signal structure and on the incoherence between the sampling and sparsifying bases. Tangible benefits can be obtained by exploiting (i) the *asymptotic* nature of the sparsity of most real-world signals and (ii) any *asymptotic* nature of the incoherence of the sampling and sparsifying bases. Whilst *all* sampling strategies will inherently exploit these asymptotic properties to some extent whenever they exist, a multilevel sampling scheme is particularly well suited, as introduced earlier in §3.2.

We devised such a sampling scheme, however we do not claim that it is optimal. It does however provide a good and flexible support for testing the theory. Assuming the coefficients $f \in \mathbb{C}^{N \times N}$ of a sampling orthobasis, such as the DFT, our multilevel sampling scheme divides f into n regions delimited by $n - 1$ equispaced concentric circles plus the full square, an example being shown in Figure 5. Normalizing the support of f to $[-1, 1]^2$, the circles have radius r_k with $k = 0, \ldots, n - 1$, which are given by $r_0 = m$ and



(a) $n=10$, $m=0.05$, $a=1.75$, $b=2.25$ (b) $n=100$, $m=0.05$, $a=1.75$, $b=2.4726$

Figure 5: Examples of subsampling maps at 2048×2048 that subsample $p = 15\%$ of Fourier coefficients. Left: 10 levels, right: 100 levels. The color intensity denotes the fraction p_k of random samples taken uniformly, i.e. white: 100% samples, black: 0% samples.

$r_k = k \cdot \frac{1-m}{n-1}$ for $k > 0$, where $0 \leq m < 1$ is a parameter. In each of the n regions, the fraction p_k of sampling coefficients taken with uniform probability is given by:

$$p_k = \exp\left(-\left(b \cdot \frac{k}{n}\right)^a\right), \tag{8.1}$$

where $k = 0, \ldots, n$ and $a > 0$ and $b > 0$ are parameters. This is similar to the generalized Gaussian density function. The total fraction of subsampled coefficients is $p = \sum_k p_k S_k$, where S_k is the normalized area of the kth region. Since $p_0 = 1$ and $p_k > p_{k+1}$, the first region will sample all coefficients and the remaining regions will sample a decreasing fraction.

This sampling scheme is extensible to other orthonormal bases provided that the coefficients f are reordered accordingly. For Hadamard with *sequency* ordering [70] or DCT, a quadrant of the above sampling scheme can be used without other modifications, as seen earlier in Figure 2, and also later in this section.

8.2 Resolution dependence: Fixed fraction p of samples

One of the main effects of asymptotic sparsity and asymptotic incoherence is that the success of compressed sensing is *resolution dependent*. We explain what this means below.

In some real-world applications, such as MRI, the underlying model is a continuous one, where the coefficients f are samples of the continuous (integral) Fourier transform. This is the correct model to use as it avoids the inverse crime which results from using the discrete Fourier transform (see §5) and it is an excellent fit for the new asymptotic sparsity and asymptotic incoherence theory.

Discrete models, however, e.g. those based on Hadamard sampling in fluorescence microscopy [67] or compressive imaging [45], also exhibit asymptotic sparsity and asymptotic incoherence since the sparsity basis arises as a discretization of a countable basis on a Hilbert space and, as shown in the following examples, greatly benefit from techniques that exploit the asymptotic behaviour.

For the MRI scenario, we use the novel GLPU phantom invented by Guerquin–Kern, Lejeune, Pruessmann and Unser [41] in favour of discretized models (e.g. the Shepp-Logan Phantom from MATLAB). The GLPU phantom is a so-called analytic phantom, in that it is not a rasterized image but a continuous (or *infinite-resolution*) object defined by analytic curves, e.g. Bezier curves. The code offered by the authors allows one to compute the continuous (integral) Fourier transform (as in a real life MRI scenario) of the GLPU phantom, for any spectral point and hence any resolution, to avoid the inverse crime.

Fixed subsampling fraction. The first important resolution dependence effect is that *regardless of the sampling basis and subsampling scheme used*, when maintaining the same fraction of samples and the same subsampling scheme, the quality of the reconstruction will increase as the resolution increases when compared to the full sampled version. This is because signals are typically increasingly sparse at high resolution levels. If the incoherence also increases asymptotically, e.g. owing to a pertinent choice of the sampling basis, then additional quality gains can be obtained by using a multilevel sampling scheme. This incoherence aspect will be investigated in more detail in §8.6.

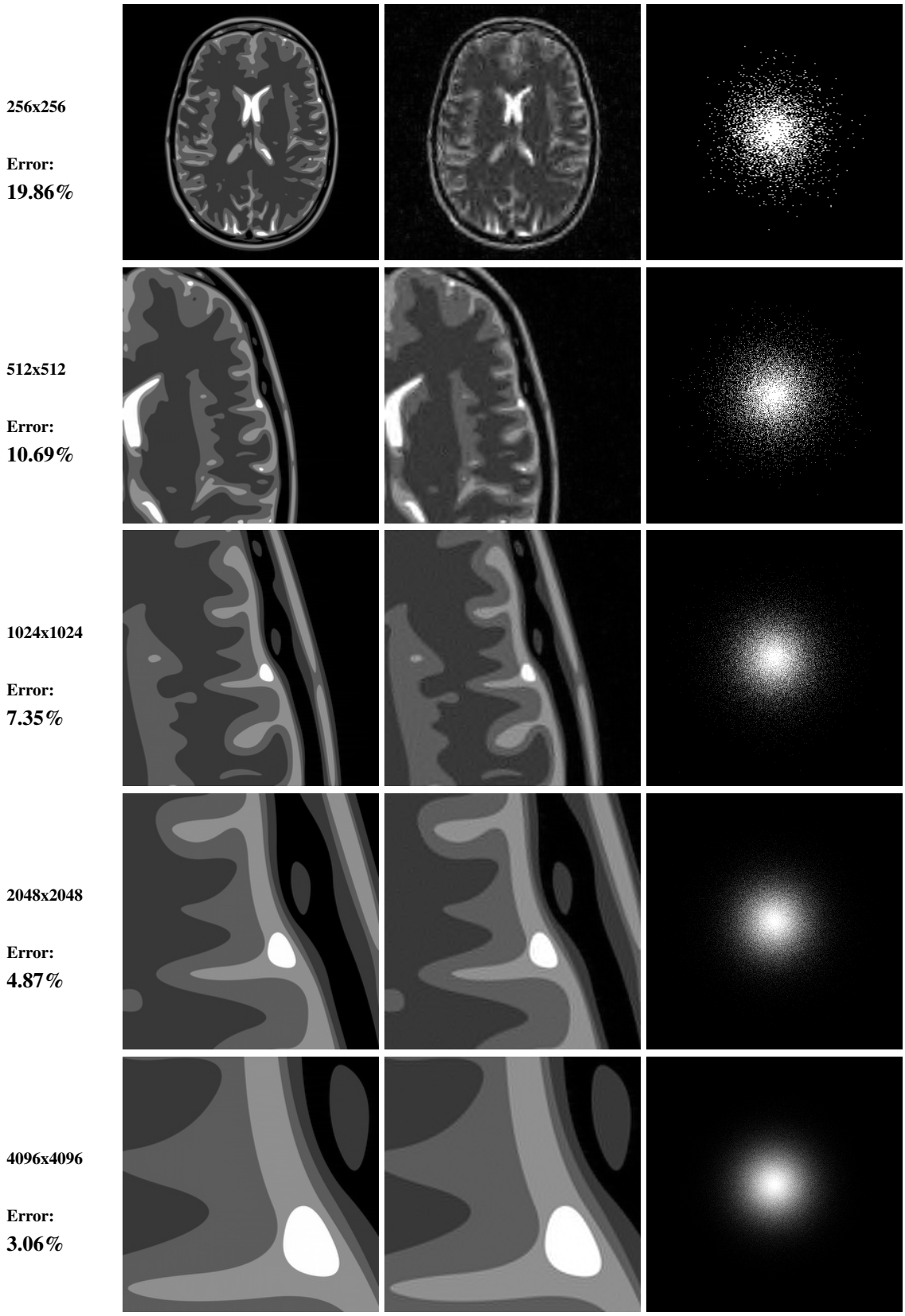


Figure 6: Multi-level subsampling of 5% Fourier coefficients using (8.1) with fixed $n = 100$, $a = 1.75$, $b = 4.125$. The left column (full sampled) and center column (subsampled) are crops of 256×256 pixels. The right column shows the uncropped subsampling map. The error shown is the relative error between the subsampled and full sampled versions.

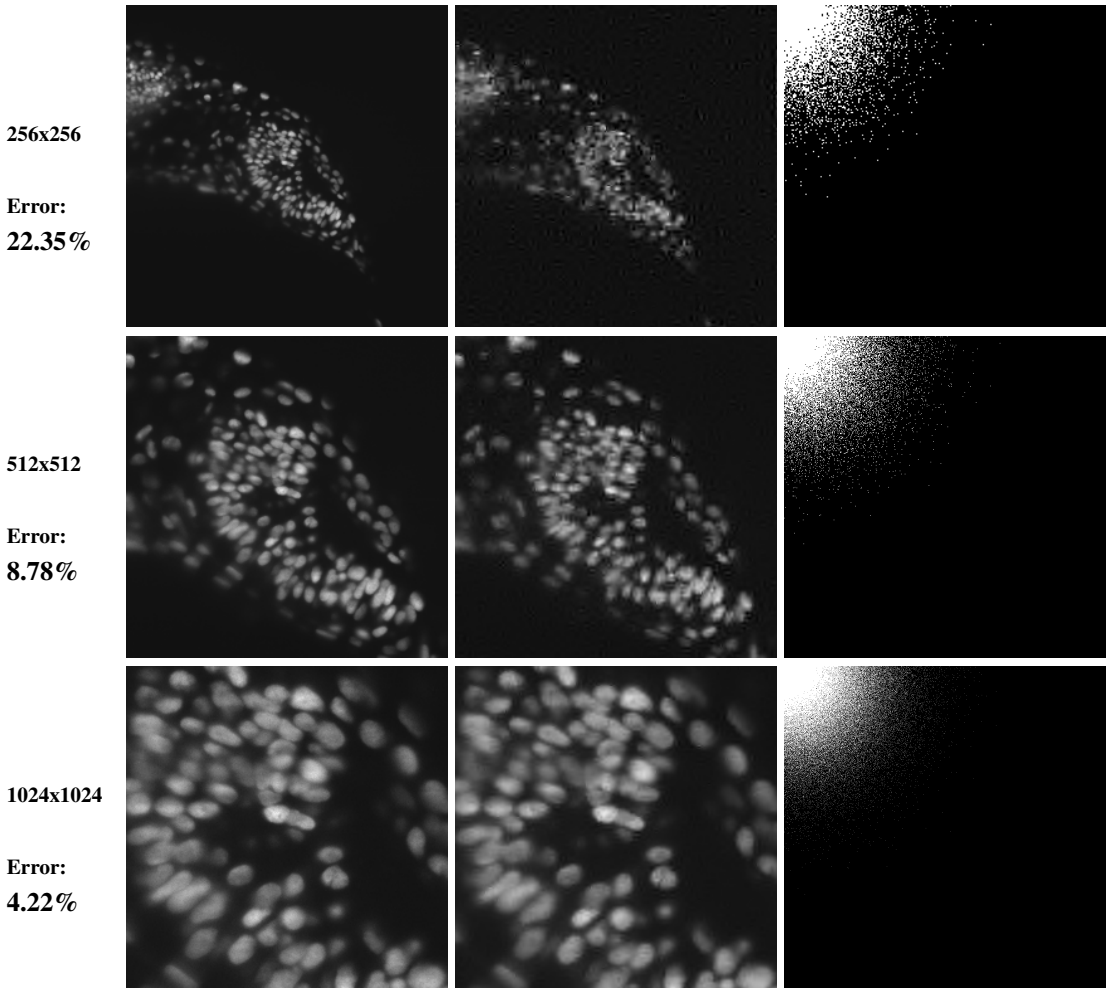


Figure 7: Multi-level subsampling of 6.25% Hadamard coefficients using (8.1) with fixed $n = 50$, $a = 2$, $b = 4.55$. The left column (full sampled) and center column (subsampled) are crops of 256×256 pixels. The right column shows the uncropped subsampling map. The error shown is the relative error between the subsampled and full sampled versions.

This effect can be seen in Figure 6 where we used the continuous GLPU phantom model. Here we fixed the parameters m, a, b in (8.1) and subsampled a constant fraction $p = 5\%$ of Fourier coefficients at increasing image resolutions from 256×256 to 4096×4096, reconstructing in the Daubechies 4 wavelet basis. The asymptotic sparsity of the wavelet coefficients allows a markedly better quality of reconstruction as the resolution increases when compared to the full sampled version.

The same effect can be observed in Figure 7, a real-life fluorescence microscope image and a discrete Hadamard model. Here we sample a constant fraction $p = 6.25\%$ of Hadamard coefficients and reconstruct in the Daubechies 6 basis. We observe the same effect, the quality of reconstruction getting substantially better as the resolution doubles.

The last example shown in Figure 8 takes a constant fraction of 6.25% using a random Bernoulli sensing matrix instead of an orthonormal basis. As stated above, this too exploits the asymptotic behaviour, and an improved reconstruction is observed as the resolution doubles. However, while the image is asymptotically sparse in wavelets, the incoherence in this case is no longer asymptotic due to the uniformly random nature of the sensing matrix – random Bernoulli is known to provide universality and uniform incoherence [36]. The lack of asymptotic incoherence, as explained later in §8.6, yields an obvious decrease in reconstruction quality in this case when compared to multilevel Hadamard from Figure 7.

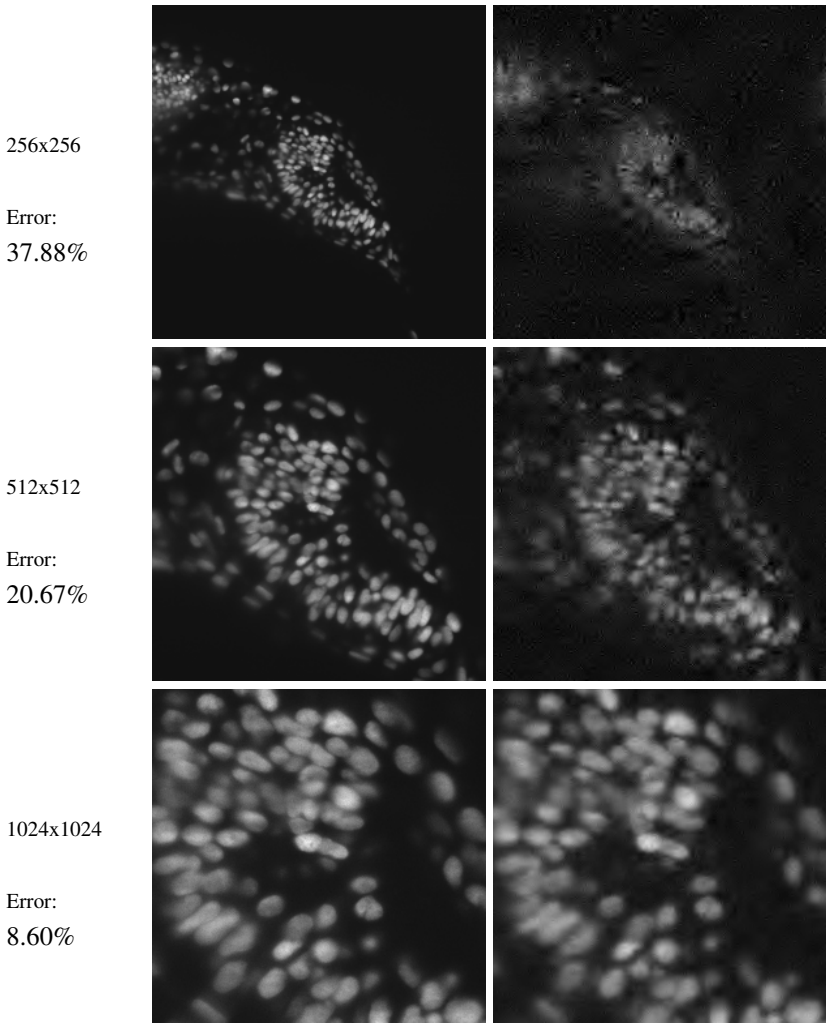


Figure 8: Random Bernoulli with 6.25% subsampling. The left column (full sampled) and center column (subsampled) are crops of 256×256 pixels. The error shown is the relative error between the subsampled and full sampled versions.

8.3 Resolution dependence: Fixed number of samples

The previous result of resolution dependence with a fixed fraction p is primarily due to asymptotic sparsity and asymptotic incoherence, but is partly also due to the fact that a fixed fraction p does mean more coefficients being sampled as the resolution increases.

A more spectacular result of asymptotic sparsity and asymptotic incoherence is obtained, this time, by fixing the number of coefficients being sampled, instead of the fraction p. This was done in Figure 9, where the same number of $512^2 = 262144$ Fourier coefficients was sampled in all of the following five reconstructions: (a) full sampled version of 512×512 pixels, (b) linear reconstruction of the subsampled 2048×2048 version by zero-padding the first 512×512 coefficients, (c) nonlinear reconstruction (i.e. l^1 minimization) into Daubechies 4 wavelets from the first 512×512 coefficients, (d) linear reconstruction from 512^2 Fourier coefficients at 2048×2048 taken using (8.1), and (e) nonlinear reconstruction into Daubechies 4 wavelets from the same multilevel subsampled coefficients from (d).

The higher resolution opens up higher-order regions of wavelet coefficients which are mostly sparse, and higher-order regions of incoherence between sinosoids and wavelets (see Figure 4). When using a nonlinear reconstruction, this asymptotic effect can be fruitfully exploited with a multilevel sampling scheme, which spreads the same number of samples across a wider range to sample from coherent regions and reconstruct finer details to a much clearer extent even in the presence of noise. Other sampling strategies (e.g. half-half)

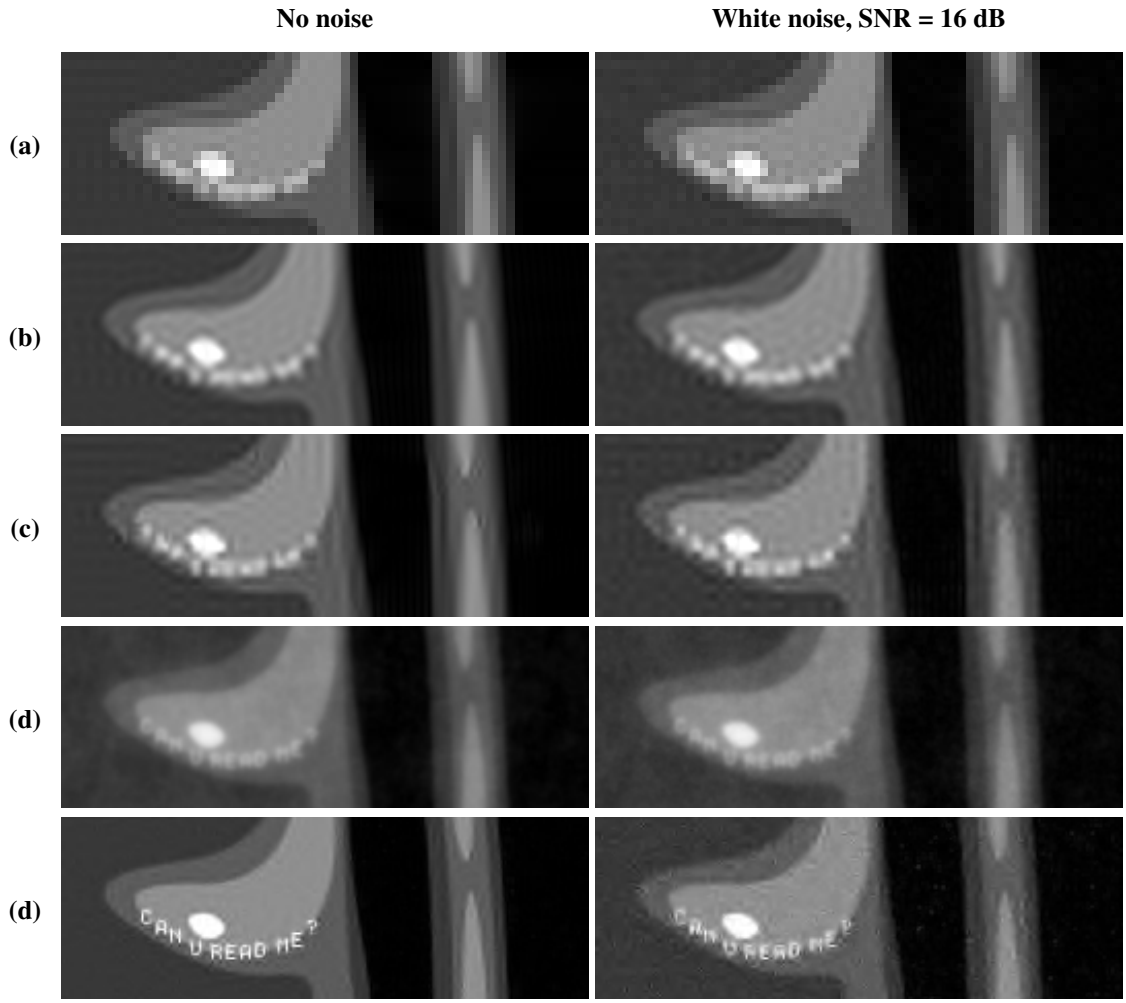


Figure 9: Subsampling a fixed number of $512^2 = 262144$ Fourier coefficients in five scenarios. **(a)** 512×512 full sampled reconstruction. **(b)** 2048×2048 linear reconstruction from the first $512\times512 = 262144$ Fourier coefficients (zero padded). **(c)** 2048×2048 reconstruction into Daubechies 4 from the first $512\times512 = 262144$ Fourier coefficients (zero padded). **(d)** 2048×2048 linear reconstruction from $512^2 = 262144$ Fourier coefficients taken with the multilevel scheme (8.1) with $n = 100$, $m = 0.05$, $a = 1.25$, $b = 4.2539$. **(e)** 2048×2048 reconstructed into Daubechies 4 from the same $512^2 = 262144$ Fourier coefficients from (d).

will also benefit from sampling at higher resolutions, provided measurements are sufficiently spread out, but a multilevel sampling strategy will provide near optimal results.

Figure 10 shows the same effect when using Hadamard as the sampling basis, subsampling the same fixed number of $512^2 = 262144$ Hadamard coefficients.

Effectively, by simply going higher in resolution (in these examples, further in the Fourier or Hadamard spectra), one can recover a signal much closer to the exact one, yet taking the same amount of measurements. Similarly, by simply going higher in the resolution one can obtain the same quality of reconstruction, yet taking less measurements.

8.4 Storage and speed

Much of the CS literature relies on the usage of random sensing matrices, typically of Bernoulli or Gaussian type, with one important reason being that these matrices provide universality (see next section §8.5). Some real-world applications proposed their use as well, e.g. the single pixel camera [68].

The Achilles' heel of these random matrices is the need for (large) storage and implicitly the lack of fast transforms. This limits the maximum image size that can be used and yield slow recovery. For example, a 1024×1024 experiment involving subsampling 12.5% of the rows of a random Gaussian matrix would

Figure 10: Subsampling a fixed number of $512^2 = 262144$ Hadamard coefficients. *(left)* 2048×2048 linear reconstruction from the first 262144 Hadamard coefficients (zero padding). *(right)* 2048×2048 reconstruction into Daubechies 12 from 262144 Hadamard coefficients taken with the multilevel scheme (8.1) with $n = 50$, $m = 0.05$, $a = 1.25$, $b = 4.5458$.

require 2 TB of free memory to store it (assuming 8 byte floating point representation), which will be read and multiplied many times during the recovery process, severely slowing it down.

A low maximum resolution is an important limiting factor not only for computations. At low resolutions, as we have seen in §8.2 and particularly in §8.3, the asymptotic sparsity of natural images in standard sparsifying transformations such as wavelets has usually not kicked in. Thus, compressed sensing can yield only marginal benefits over more classical reconstructions.

Some solutions to the storage and speed problems which typically try to preserve universality exist, such as pseudo-random permutations of the columns of Hadamard or block Hadamard matrices [37, 45], Kronecker products of random matrix stencils [32], or even orthogonal transforms such as the Sum-To-One (STOne) matrix [38]. The STOne matrix has a fast transform and does not need to be stored, whereas solutions that employ randomization address the storage and speed problems only to an extent.

It is known that for sufficiently large problem sizes, continuous random matrices (e.g. Gaussian) and discrete random matrices (e.g. Bernoulli) share the same statistics. With the new theory in this paper, a solution to the storage and speed problems is to replace random sensing matrices with an orthogonal matrix, like the Hadamard matrix, and then subsample it using a multilevel scheme such as (8.1). Other matrices also work, e.g. DCT, DFT etc, but the Hadamard matrix can be more easily implemented in certain physical

devices, due to its binary structure (see §8.5). These provide asymptotic incoherence with wavelets and other common sparsifying transformations, and have $\mathcal{O}(N\log(N))$ fast transforms. However, they do not provide universality. But is universality actually a desirable feature in a sensing matrix given that the signal to be recovered is highly structured?

8.5 Structure vs. Universality and RIP

Traditionally, random matrices have been considered convenient in CS, since, amongst other properties, they provide universality. A random matrix $A \in \mathbb{C}^{m\times N}$ is universal if for any isometry $U \in \mathbb{C}^{N\times N}$, the matrix AU satisfies the RIP with high probability. Random Bernoulli or Gaussian matrices have such a property, and for this reason, they have become popular choices as sensing matrices. A common choice for 2D signals is $U = V_{\mathrm{dwt}}^{-1}$, the inverse wavelet transform.

Compressive imaging [68, 45] is an important example where universal sensing matrices have frequently been used. The idea behind the sensing device is that measurements are taken using a sensing matrix with only 1 and -1 as values, i.e. $y = Ax$, where $x \in \mathbb{C}^N$ is the signal to be recovered, $y \in \mathbb{C}^m$ with $m \leq N$ is the vector of measurements and $A \in \{-1, 1\}^{m\times N}$ is the sensing matrix[1]. All matrices with only two values ($\{-1, 1\}$ can be obtained from a simple transformation) fit this setup: random Bernoulli matrices, Hadamard matrices and derivations, the STOne matrix etc.

Figure 11 shows the result of an interesting experiment. As sensing matrices we used a random Bernoulli, a random Gaussian and STOne. As sparsifying bases we used wavelets, discrete cosine, and Hadamard. We then performed the flip test (§2.3). We tested a variety of different images, resolutions and subsampled fractions, the result being the same as in Figure 11: the three sensing matrices all behave extremely similarly and the RIP holds almost exactly, being a strong indicator that the STOne matrix also provides universality like the random Bernoulli and Gaussian matrices.

But is universality a feature one wants in a sensing matrix when the signal of interest is structured? With universal sensing matrices, there is no room to exploit any extra structure the signal may have. The underlying reason is explained in the next subsection §8.6.

An alternative to A is to use a non-universal sensing matrix such as the Hadamard matrix, U_{Had}. Note that $U_{\mathrm{Had}}V_{\mathrm{dwt}}^{-1}$ is completely coherent (regardless of the wavelet used), but is asymptotically incoherent (see Figure 4) and thus perfectly suitable for a multilevel sampling scheme which can exploit any asymptotic sparsity typically prevalent in real world signals.

This is precisely what we see in Figure 12 (the top first two images are repeated from Figure 11), which shows a comparison between two universal sensing matrices, random Bernoulli and STOne, subsampled uniformly at random (STOne was also subsampled semi-randomly as specified in [38] which gave very similar results), and Hadamard subsampled using the multilevel scheme (8.1).

Given that most real world signals have an asymptotic sparsity structure in some basis, at least as far as imaging is concerned, then it is desirable to use a sensing matrix which provides asymptotic incoherence, which in turn allows the asymptotic sparsity of the signal to be exploited using structured sampling strategies such as multilevel sampling.

An important aspect is that, unlike compressive imaging where one has significant freedom to design the sensing matrix (usually having binary entries, typically -1,1), many real-world problems such as MRI, CT and others will impose the sensing matrix, i.e. there is no freedom design or choose sensing matrices; random matrices are simply not applicable. Luckily, in many such applications the imposed sensing operators are highly non-universal and asymptotically incoherent with popular sparsifying bases, and thus easily exploitable using multilevel sampling, as shown in our previous experiments.

8.6 Asymptotic incoherence vs. Uniform incoherence

The reasons behind the previous results are deeply rooted in the incoherence structure. Universal sensing matrices and those that are close to universal typically provide a relatively low and uniform incoherence pattern. This allows sparsity to be exploited by sampling uniformly at random. Yet, by definition, universal sensing matrices cannot exploit the distinct asymptotic sparsity structure of real-world signals when using a typical (ℓ^1 minimization) compressed sensing reconstruction.

[1] In a practical implementation, $\{0, 1\}$ may be used instead of $\{-1, 1\}$, as it corresponds to *open/closed* states of hardware elements. A post-measurement transformation is applied to provide an equivalence to using a $\{-1, 1\}^{N\times m}$ matrix.

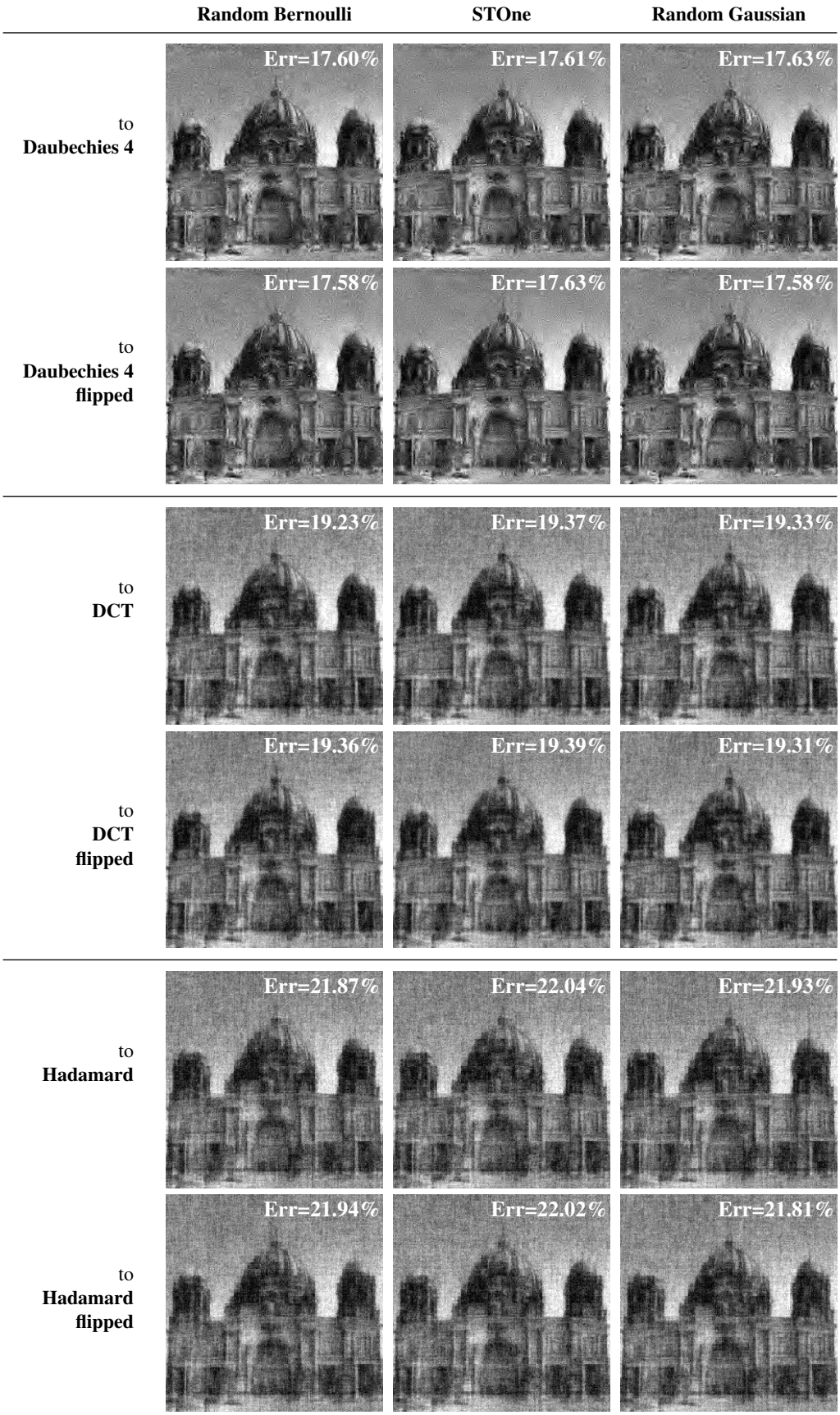
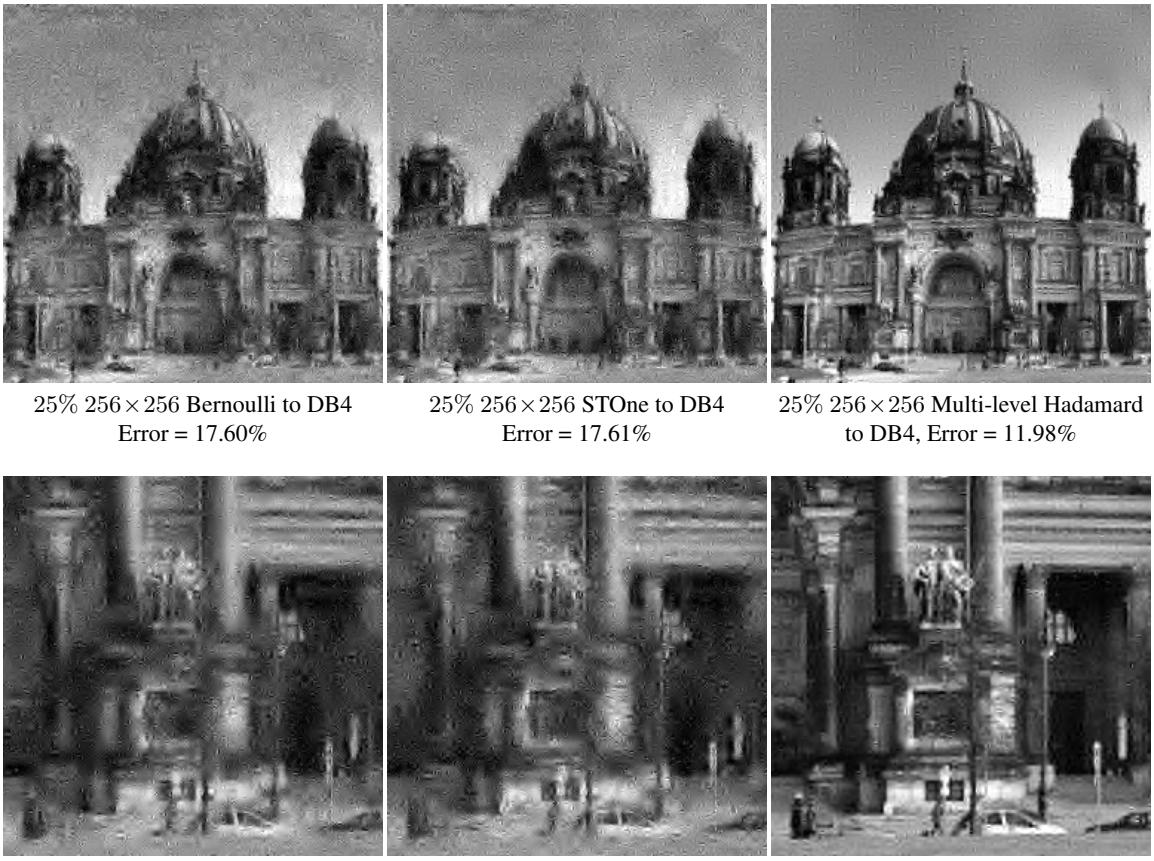


Figure 11: A 25% subsampling experiment at 256×256 with Barnoulli, Gaussian and STOne to various orthobases. See §2.3 for recovery using flipped coefficients. It is obvious that images in each row have very similar quality, and the same holds for the flipped vs. non-flipped recovery. This strongly indicates that Bernoulli, Gaussian and STOne matrices behave the same, and that *universality* and *RIP* hold for all three.



25% 256×256 Bernoulli to DB4 Error = 17.60%

25% 256×256 STOne to DB4 Error = 17.61%

25% 256×256 Multi-level Hadamard to DB4, Error = 11.98%

12.5% 1024×1024 Bernoulli to DB4, Error = 17.00%

12.5% 1024×1024 STOne to DB4 Error = 16.96%

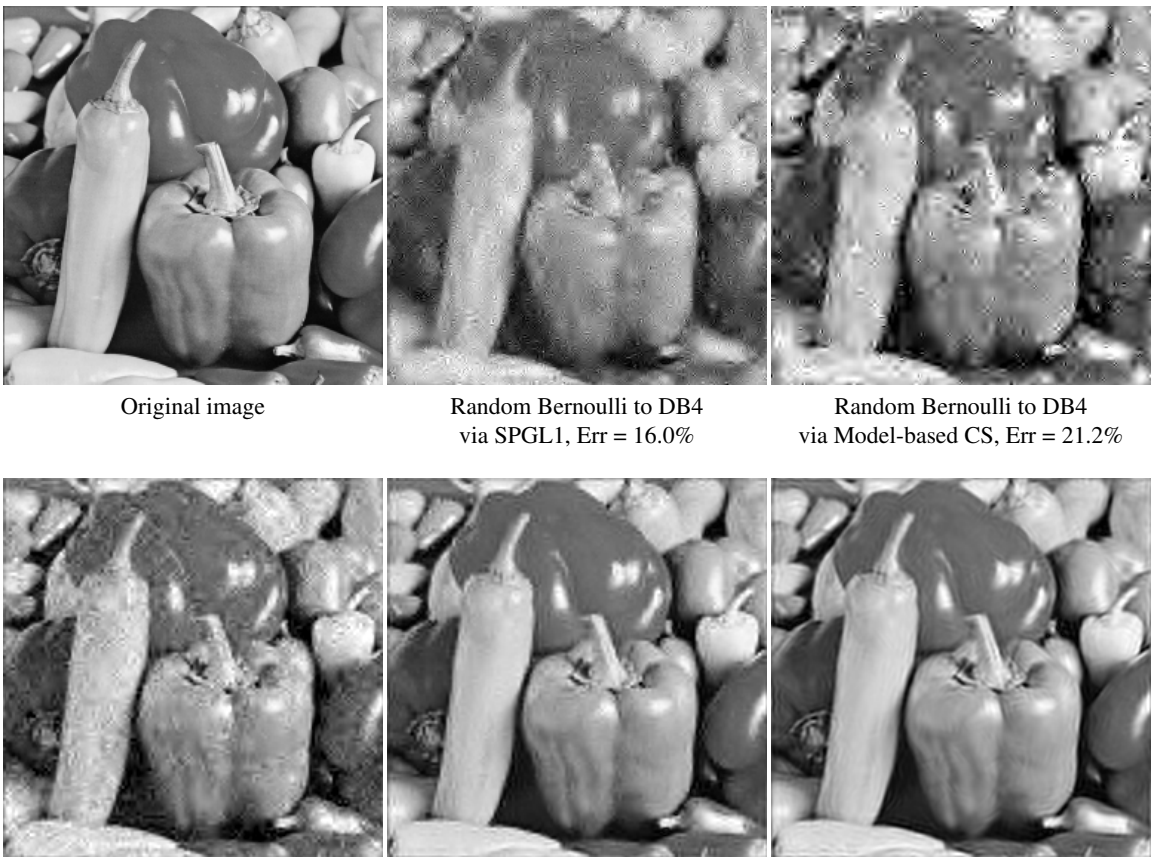
12.5% 1024×1024 Multi-level Hadamard to DB4, Error = 10.40%

Figure 12: Random Bernoulli and STOne, and Multi-level Hadamard reconstructed in Daubechies 4. The images on the bottom row are 256×256 crops of the original 1024×1024 reconstructions.

Conversely, when a sensing matrix is designed so that the coherence behaviour aligns with the sparsity pattern of the signal, one can indeed exploit such structure. In particular, a multilevel sampling scheme is likely to give good results by sampling more in the coherent regions where the signal is also typically less sparse. We remark, however, that the optimum sampling strategy is also highly dependent on the signal structure. Therefore if a particular image does not have its important coefficients in the coherent regions, one may witness an inferior reconstruction. Yet most images share a reasonably common structure, and thus we can find good all-round multilevel sampling strategies.

As seen in the previous subsection (in the case of STOne), in order to provide asymptotic incoherence, the sensing matrix should contain additional structure besides simply being non-random. Typically, sensing and sparsifying matrices that are discrete versions of integral transforms, e.g. Fourier, wavelets, etc. will provide asymptotic incoherence, but others like Hadamard will do so too. As discussed earlier, this scenario also matches many real-world applications where the sensing operators are imposed, e.g. MRI or CT.

Note on the usage of the STOne matrix. We used the STOne matrix in the examples above purely as an example of a matrix that provides a low and relatively uniform incoherence like random matrices, but which is a not a random matrix. However, it is worth noting that the STOne matrix was invented for different primary purposes than universality or performance in compressed sensing recovery of 2D images. It has a fast $\mathcal{O}(N \log N)$ transform and allows multi-scale image recovery from compressive measurements: low-resolution previews can be instantly generated by applying the fast transform on the measurements directly, and high resolution recovery is possible from the same measurements via compressed sensing solvers. In addition, it was also designed to allow efficient, real-time recovery of compressive videos when sampling in a particular manner semi-randomly. These features make the STOne matrix a versatile sensing operator.



Original image | Random Bernoulli to DB4 via SPGL1, Err = 16.0% | Random Bernoulli to DB4 via Model-based CS, Err = 21.2%

Random Bernoulli to DB4 via TurboAMP, Err = 17.5% | Multi-level Hadamard to DB4, via SPGL1, Err = 7.1% | Multi-level Hadamard to Curvelets via SPGL1, Err = 6.5%

Figure 13: Subsampling 12.5% coefficients at 256×256. The error shown is the relative error to the fully sampled image.

8.7 Structured sampling vs. Structured recovery

Until now, we discussed taking into account sparsity structure in the sampling procedure via multilevel sampling of non-universal sensing matrices. Sparsity structure can also be taken into account in the recovery algorithm. An example of such an approach is model-based compressed sensing [8], which assumes the signal is piecewise smooth and exploits the connected tree structure of its wavelet coefficients to reduce the search space of the matching pursuit algorithm [58]. Another approach is the class of message passing and approximate message passing algorithms (AMP) [9, 29], which exploit the persistence across scales structure [56] of wavelet coefficients by a modification to iterative thresholding algorithms inspired by belief propagation from graph models. This can be coupled with hidden Markov trees to model the wavelet structure, such as in the Turbo AMP algorithm [65]. For earlier work in this direction, see [42, 43].

Another approach is to assume that the actual signal – not its representation in some sparsifying basis – is sparse and random, or randomly indexed, and from a known probability distribution [76], which allows probabilistic reconstruction via spatially coupled sensing matrices (random matrices with band diagonal structure) provided their structure is linked to the signal's sparsity [49, 48, 27]. If all these assumptions are met, robust reconstruction is possible at subsampling rates close to the signal's sparsity (this subsampling bound is however sharp, the recovery failing beyond it; algorithms based on convex optimization do not exhibit this sharp effect). However, we omit this approach from our experiment, since it is not clear how it can be effectively modified to work for problems where the image – the object being sampled – is sparse in a transform domain, such is the case in many applications.

The main difference between multilevel sampling of asymptotically incoherent matrices and the approaches mentioned above is that the former incorporates the sparsity structure in the sampling strategy and

uses an unmodified CS recovery algorithm (l^1 minimization), whereas the latter use standard CS sampling strategies (universal sensing matrices, e.g. random Gaussian/Bernoulli) and exploit structure by modifying the recovery algorithm. In particular, the latter are based on universal sensing matrices with corresponding uniform incoherence patterns, an aspect discussed earlier in §8.6 and §8.5. It is thus of interest to compare the approaches.

Figure 13 shows a representative experiment from a large set of experiments that we ran. As can be seen, it points to the same conclusion: that asymptotic incoherence combined with multilevel sampling of highly non-universal sensing matrices (e.g. Hadamard, Fourier) allows structured sparsity to be better exploited than universal sensing matrices with structure being incorporated into the recovery algorithm. It is also worth restating that in many applications (e.g. MRI, CT and others) where the sensing matrix is imposed and non-universal, one cannot use the sensing matrices needed by these algorithms. Conversely, as shown in this paper, multilevel subsampling of the imposed matrix provides near-optimal recovery guarantees.

8.8 Orthobases vs. Frames

The previous subsection in Figure 13 included two results of an image reconstructed in a frame, rather than an orthonormal basis. Although it is not the purpose of this paper to investigate the usage of frames as sparsifying matrices in compressed sensing, we provide nonetheless some further results obtained using frames.

Without going into any detail, the main differences of interest to us for 2D signals between orthonormal basis and frames are that many images are known to be relatively sparser in frames such as curvelets [11, 13], shearlets [20, 21, 50] or contourlets [26, 59] than in orthonormal basis such as wavelets or DCT. Note that our results provide clear experimental verifications of the improvements offered by the aforementioned frames – in particular, the recently-introduced shearlets – at practical resolution levels.

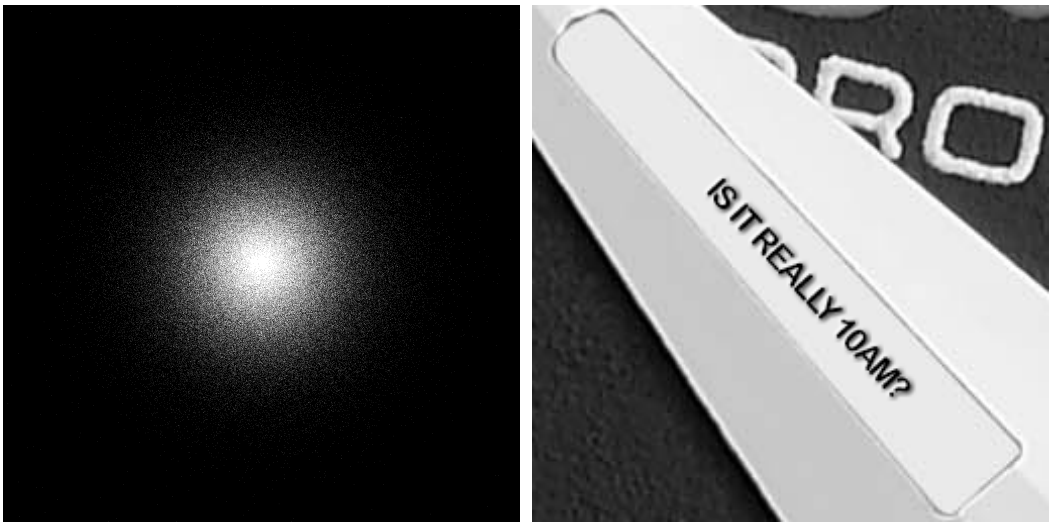
Figure 14 shows the recovery of a high resolution image with fine details, and one can observe that indeed the frames manage to yield a higher reconstruction quality than wavelets. This is an encouraging result and an avenue worth investigating further theoretically and practically.

9 Conclusions

In this paper we have introduced a new framework for compressed sensing. This generalizes the existing theory and shows that compressed sensing is possible under greatly relaxed conditions. The importance of this is threefold. First, we have given previously-lacking mathematical credence to the abundance of numerical studies showing the success of compressed sensing in inverse problems such as MRI. Second, in showing that compressed sensing is possible in the presence of only asymptotic incoherence, our theory raises the possibility of substantially greater flexibility in the future design of sensing mechanisms. Third, we have shown the importance of exploiting not just sparsity but also structure so as to get as high quality reconstructions as possible. In particular, this can be achieved in a computationally efficient (and physically realizable) manner using Hadamard matrices with multilevel subsampling, for example, in compressive imaging.

Note that structure in compressed sensing has been discussed in many previous works; see, for example [28, 72] for considering the wavelet structure in the sampling strategy, [17, 75] for design of sensing matrices given known structure of the signal to be recovered, and [33] for general discussions on structure in compressed sensing. As discussed, however, existing structure-exploiting algorithms are typically based on universal sensing matrices, and leverage structure in the recovery algorithm. Conversely, our approach exploits structure in the sampling strategy using highly non-universal operators, and, as we have shown, this leads to substantial improvements. This begs the question: are universality and RIP what we want in practice? This is an interesting topic for future consideration.

As discussed, an important application of our work lies in MRI. Our main result has provided the first comprehensive explanation for the success of compressed sensing in MRI. We note that there have been several prior works on this topic. These include [47], which analyzed the case of bivariate Haar wavelets with an inverse power law sampling density, and [10] where block sampling strategies were analyzed, and in particular, Shannon wavelets with horizontal line sampling. Both works differ from ours in that they consider only sparsity, and thus do not adhere to the conclusions of the flip test. Having said this, the results of [10] are closer to the types of sampling strategies that can be implemented in actual MRI problems, where one must sample along continuous curves, due to the physics of the scanner. A objective of future work is to extend our results to incorporate both structured sparsity and realistic contours. We note also that [47]



Subsampling map

Original image

Wavelets (DB4)

Curvelets

Contourlets

Shearlets

Figure 14: Multi-level subsampling of 6.25% of DFT coefficients at 2048×2048 of the same image used in Figure 10. All images are crops of 256×256 of the full 2048×2048 version, except the subsampling map.

provides recovery guarantees for TV minimization; an important and popular strategy in imaging. This is another direction for future investigations.

We have concluded in our work that the optimal sampling strategy is dependent on the signal structure. Further work is required to determine such strategies in a rigorous empirical manner for important classes of images. We expect our main theorems to give important insights into these investigations, such as how many levels to choose, how to choose their relative sizes, etc. One conclusion of our work, however, is that approaches to design optimal sampling densities based solely on minimizing coherences (i.e. not taking into account asymptotic sparsity) may be of little use in practice unless they are trained on large families of images having similar structures (e.g. brain images).

As discussed and highlighted in our numerics, asymptotic sparsity is not only relevant for wavelets. Any approximation system whose power lies in nonlinear, as opposed to linear, approximation will give rise to asymptotically sparse representations. Such systems include curvelets [11, 13], contourlets [26, 59] or shearlets [20, 21, 50], all of which find application in inverse problems and, as we have shown experimentally, carry advantages in compressive imaging over wavelet-based approaches. An immediate objective of future work is to extend our analysis for the Fourier/wavelets case to these more exotic function systems.

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