# Trajectory of Alternating Direction Method of Multipliers and Adaptive Acceleration 

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#### Abstract

The alternating direction method of multipliers (ADMM) is one of the most widely used first-order optimisation methods in the literature owing to its simplicity and efficiency. Over the years, different efforts are made to improve the method, such as the inertial technique. By studying the geometric properties of ADMM, we discuss the limitations of current inertial accelerated ADMM and then present and analyse an adaptive acceleration scheme for ADMM. Numerical experiments on problems arising from image processing, statistics and machine learning demonstrate the advantages of the proposed algorithm.


## 1 Introduction

Consider the following constrained and composite optimisation problem

$$
\begin{equation*}
\min _{x \in \mathbb{R}^{n}, y \in \mathbb{R}^{m}} R(x)+J(y) \quad \text { such that } \quad A x+B y=b, \tag{P}
\end{equation*}
$$

where the following basic assumptions are imposed
$(\mathcal{A . 1}) R \in \Gamma_{0}\left(\mathbb{R}^{n}\right)$ and $J \in \Gamma_{0}\left(\mathbb{R}^{m}\right)$ are proper convex and lower semi-continuous.
$(\mathcal{A . 2}) A: \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}$ and $B: \mathbb{R}^{m} \rightarrow \mathbb{R}^{p}$ are injective linear operators.
$(\mathcal{A} .3) \operatorname{ri}(\operatorname{dom}(R) \cap \operatorname{dom}(J)) \neq \emptyset$, and the set of minimisers is non-empty.
Over the past years, problem $(\mathcal{P})$ has attracted a great deal of interests as it covers many important problems arising from data science, machine learning, statistics and image processing, etc.; See Section 5 for examples. In the literature, different solvers are proposed to handle the problem, among them the alternating direction method of multipliers (ADMM) is the most prevailing one.

ADMM was first proposed in [20] and becomes increasingly popular recently owing to [10]. The Lagrangian associated to $(\mathcal{P})$ reads

$$
\mathcal{L}(x, y ; \psi) \stackrel{\text { def }}{=} R(x)+J(y)+\langle\psi, A x+B y-b\rangle,
$$

and the augmented Lagrangian then simply is:

$$
\mathcal{L}_{\gamma}(x, y ; \psi) \stackrel{\text { def }}{=} \mathcal{L}(x, y ; \psi)+\frac{\gamma}{2}\|A x+B y-b\|^{2},
$$

where $\gamma>0$. To find a saddle-point of $\mathcal{L}(x, y ; \psi)$, ADMM applies the following iteration

$$
\begin{align*}
x_{k} & =\operatorname{argmin}_{x \in \mathbb{R}^{n}} R(x)+\frac{\gamma}{2}\left\|A x+B y_{k-1}-b+\frac{1}{\gamma} \psi_{k-1}\right\|^{2} \\
y_{k} & =\operatorname{argmin}_{y \in \mathbb{R}^{m}} J(y)+\frac{\gamma}{2}\left\|A x_{k}+B y-b+\frac{1}{\gamma} \psi_{k-1}\right\|^{2}  \tag{1.1}\\
\psi_{k} & =\psi_{k-1}+\gamma\left(A x_{k}+B y_{k}-b\right) .
\end{align*}
$$

[^0]Define a new point $z_{k} \stackrel{\text { def }}{=} \psi_{k-1}+\gamma A x_{k}$, then we can rewrite ADMM iteration (1.1) as

$$
\begin{align*}
x_{k} & =\operatorname{argmin}_{x \in \mathbb{R}^{n}} R(x)+\frac{\gamma}{2}\left\|A x-\frac{1}{\gamma}\left(z_{k-1}-2 \psi_{k-1}\right)\right\|^{2} \\
z_{k} & =\psi_{k-1}+\gamma A x_{k} \\
y_{k} & =\operatorname{argmin}_{y \in \mathbb{R}^{m}} J(y)+\frac{\gamma}{2}\left\|B y+\frac{1}{\gamma}\left(z_{k}-\gamma b\right)\right\|^{2}  \tag{1.2}\\
\psi_{k} & =z_{k}+\gamma\left(B y_{k}-b\right) .
\end{align*}
$$

For the rest of the paper, we will consider the above four-point formulation.

### 1.1 Contributions

The contribution of our paper is threefold. First, for the sequence $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ of (1.2), we prove that it has two different types of trajectory:

- When both $R, J$ are non-smooth functions, under the assumption that they are partly smooth (see Definition 2.1), we show that the eventual trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is approximately a spiral which can be characterised precisely if $R, J$ are moreover locally polyhedral.
- When at least one of $R, J$ is smooth, we show that under properly chosen $\gamma$, the eventual trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is a straight line.
Then, based on trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$, we discuss the limitations of the current combination between ADMM and inertial acceleration technique. In Section 3, we distinguish the situations where inertial acceleration will work and when it fails. More precisely: inertial technique will work if the trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is or close to a straight line, and will fail if the trajectory is a spiral.

Our core contribution is an adaptive acceleration for ADMM, which is inspired by the trajectory of ADMM and dubbed $\mathrm{A}^{3} \mathrm{DMM}$. The limitation of inertial technique, particularly its failure, implies that the right acceleration scheme should be able to follow the trajectory of the sequence. In Section C, we propose an adaptive linear prediction scheme for accelerating ADMM which is able to following the trajectory of the generated sequence. Our proposed $A^{3} D M M$ belongs to the realm of extrapolation method, and provides an alternative interpretation for polynomial extrapolation methods such as Minimal Polynomial Extrapolation (MPE) [13] and Reduced Rank Extrapolation (RRE) [18, 29].

### 1.2 Related works

Over the past decades, owing to the tremendous success of inertial acceleration [31, 8], the inertial technique has been widely adapted to accelerate other first-order algorithms. In the realm of ADMM, related work can be found in [32, 22, 19], either from proximal point algorithm perspective or continuous dynamical system. However, to ensure that inertial acceleration works, strong assumptions are imposed on $R, J$ in ( $\mathcal{P}$ ), such as smooth differentiability or strong convexity. When it comes to general non-smooth problems, these works will fail to provide acceleration.

For more generic acceleration techniques, there are extensive works in numerical analysis on the topic of convergence acceleration for sequences. Given an arbitrary sequence $\left\{z_{k}\right\}_{k \in \mathbb{N}} \subset \mathbb{R}^{n}$ with limit $z^{\star}$, finding a transformation $\mathcal{E}_{k}:\left\{z_{k-j}\right\}_{j=1}^{q} \rightarrow \bar{z}_{k} \in \mathbb{R}^{n}$ such that $\bar{z}_{k}$ converges faster to $z^{\star}$. In general, the process by which $\left\{z_{k}\right\}$ is generated is unknown, $q$ is chosen to be a small integer, and $\bar{z}_{k}$ is referred to as the extrapolation of $z_{k}$. Some of the best known examples include Richardson's extrapolation [34], the $\Delta^{2}$-process of Aitken [2] and Shank's algorithm [36]. We refer to [11, 12, 37] and references therein for a detailed historical perspective on the development of these techniques. Much of the works on the extrapolation of vector sequences was initiated by Wynn [42] who generalized the work of Shank to vector sequences. In the appendix, the formulation of some of these methods are provided. In particular, minimal polynomial extrapolation (MPE) [13] and Reduced Rank Extrapolation (RRE) [18, 29] (which is also a variant of Anderson acceleration developed independently in [4]), which are particularly relevant to this present work (see Section 4.2).

More recently, there has been a series of work on a regularized version of RRE stemming from [35]. We remark however the regularization parameter in these works rely on a grid search based on objective function,
their applicability to the general ADMM setting is unclear.
Notations Denote $\mathbb{R}^{n}$ a $n$-dimensional Euclidean space equipped with scalar product $\langle\cdot, \cdot\rangle$ and norm $\|\cdot\|$. Id denotes the identity operator on $\mathbb{R}^{n} . \Gamma_{0}\left(\mathbb{R}^{n}\right)$ denotes the class of proper convex and lower-semicontinuous functions on $\mathbb{R}^{n}$. For a nonempty convex set $S \subset \mathbb{R}^{n}$, denote $\operatorname{ri}(S)$ its relative interior, $\operatorname{par}(S)$ the smallest subspace parallel to $S$ and $\mathcal{P}_{S}$ the projection operator onto $S$. The sub-differential of a function $R \in \Gamma_{0}\left(\mathbb{R}^{n}\right)$ is defined by $\partial R(x) \stackrel{\text { def }}{=}\left\{g \in \mathbb{R}^{n} \mid R\left(x^{\prime}\right) \geq R(x)+\left\langle g, x^{\prime}-x\right\rangle, \forall x^{\prime} \in \mathbb{R}^{n}\right\}$. The spectral radius of a matrix $M$ is denoted by $\rho(M)$.

## 2 Trajectory of ADMM

In this section, we discuss the trajectory of the sequence $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ generated by ADMM based on the concept "partial smoothness" which was first introduced in [24].

### 2.1 Partial smoothness

Let $\mathcal{M} \subset \mathbb{R}^{n}$ be a $C^{2}$-smooth submanifold, denote $\mathcal{T}_{\mathcal{M}}(x)$ the tangent space of $\mathcal{M}$ at a point $x \in \mathcal{M}$.
Definition 2.1 (Partly smooth function [24]). A function $R \in \Gamma_{0}\left(\mathbb{R}^{n}\right)$ is partly smooth at $\bar{x}$ relative to a set $\mathcal{M}_{\bar{x}}$ if $\partial R(\bar{x}) \neq \emptyset$ and $\mathcal{M}_{\bar{x}}$ is a $C^{2}$ manifold around $\bar{x}$, and moreover

Smoothness $R$ restricted to $\mathcal{M}_{\bar{x}}$ is $C^{2}$ around $\bar{x}$.
Sharpness The tangent space $\mathcal{T}_{\mathcal{M}_{\bar{x}}}(\bar{x})=\operatorname{par}(\partial R(\bar{x}))^{\perp}$.
Continuity The set-valued mapping $\partial R$ is continuous at $x$ relative to $\mathcal{M}_{\bar{x}}$.
The class of partly smooth functions at $\bar{x}$ relative to $\mathcal{M}_{\bar{x}}$ is denoted as $\operatorname{PSF}_{\bar{x}}\left(\mathcal{M}_{\bar{x}}\right)$. Popular examples of partly smooth functions can be found in [25, Chapter 5]. Loosely speaking, a partly smooth function behaves smoothly as we move along $\mathcal{M}_{\bar{x}}$, and sharply if we move transversal to it.

### 2.2 Trajectory of ADMM

The iteration of ADMM is non-linear in general owing to the non-smoothness and non-linearity of $R$ and $J$. However, if they are partly smooth, the local $C^{2}$-smoothness allows us to linearise the ADMM iteration, and hence enables us to study the trajectory of sequence generated by the method. We denote $\left(x^{\star}, y^{\star}, \psi^{\star}\right)$ a saddlepoint of $\mathcal{L}(x, y ; \psi)$ and let $z^{\star}=\psi^{\star}+\gamma A x^{\star}$.

To discuss the trajectory of ADMM, we rely on sequence $\left\{z_{k}\right\}_{k \in \mathbb{N}}$. Define $v_{k} \stackrel{\text { def }}{=} z_{k}-z_{k-1}$ and $\theta_{k} \stackrel{\text { def }}{=}$ $\arccos \left(\frac{\left\langle v_{k}, v_{k-1}\right\rangle}{\left\|v_{k}\right\|\left\|v_{k-1}\right\|}\right)$ the angle between $v_{k}, v_{k-1}$. We use $\left\{\theta_{k}\right\}_{k \in \mathbb{N}}$ to characterise the trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$. Given $\left(x^{\star}, y^{\star}, \psi^{\star}\right)$, the first-order optimality condition entails $-A^{T} \psi^{\star} \in \partial R\left(x^{\star}\right)$ and $-B^{T} \psi^{\star} \in \partial J\left(y^{\star}\right)$, below we impose

$$
\begin{equation*}
-A^{T} \psi^{\star} \in \operatorname{ri}\left(\partial R\left(x^{\star}\right)\right) \quad \text { and } \quad-B^{T} \psi^{\star} \in \operatorname{ri}\left(\partial J\left(y^{\star}\right)\right) \tag{ND}
\end{equation*}
$$

Both $R, J$ are non-smooth Suppose $R \in \operatorname{PSF}_{x^{\star}}\left(\mathcal{M}_{x^{\star}}^{R}\right), J \in \operatorname{PSF}_{y^{\star}}\left(\mathcal{M}_{y^{\star}}^{J}\right)$ are partly smooth, denote $T_{x^{\star}}^{R}, T_{y^{\star}}^{J}$ the tangent spaces of $\mathcal{M}_{x^{\star}}^{R}, \mathcal{M}_{y^{\star}}^{J}$ at $x^{\star}, y^{\star}$. Let $A_{R} \stackrel{\text { def }}{=} A \circ \mathcal{P}_{T_{x^{\star}}^{R}}, B \stackrel{\text { def }}{=} B \circ \mathcal{P}_{T_{y^{\star}}^{J}}$ and $T_{A_{R}}, T_{B_{J}}$ be the range of $A_{R}, B_{J}$ respectively. Denote $\left(\alpha_{j}\right)_{j=1, \ldots}$ the Principal angles (see Section A. 2 in the appendix for definition) between $T_{A_{R}}, T_{B_{J}}$, and let $\alpha_{F}, \alpha^{\prime}$ be the smallest and 2 nd smallest of $\alpha_{j}$ which are yet larger than 0 .

Theorem 2.2. For problem $(\mathcal{P})$ and ADMM iteration (1.1), assume that conditions $(\mathcal{A} .1)-(\mathcal{A} .3)$ are true, then $\left(x_{k}, y_{k}, \psi_{k}\right)$ converges to a saddle point $\left(x^{\star}, y^{\star}, \psi^{\star}\right)$ of $\mathcal{L}(x, y ; \psi)$. Suppose that $R \in \operatorname{PSF}_{x^{\star}}\left(\mathcal{M}_{x^{\star}}^{R}\right), J \in$ $\operatorname{PSF}_{y^{\star}}\left(\mathcal{M}_{y^{\star}}^{J}\right)$ and condition (ND) holds, then
(i) There exists a matrix $M_{\mathrm{ADMM}}$ such that $v_{k}=M_{\mathrm{ADMM}} v_{k-1}+o\left(\left\|v_{k-1}\right\|\right)$ holds for all $k$ large enough.
(ii) If moreover, $R, J$ are locally polyhedral around $x^{\star}, y^{\star}$, then $v_{k}=M_{\mathrm{ADMM}} v_{k-1}$ with $M_{\mathrm{ADMM}}$ being normal and having eigenvalues of the form $\cos \left(\alpha_{j}\right) e^{ \pm \mathrm{i} \alpha_{j}}$, and $\cos \left(\theta_{k}\right)=\cos \left(\alpha_{F}\right)+O\left(\eta^{2 k}\right)$ with $\eta=\cos \left(\alpha^{\prime}\right) / \cos \left(\alpha_{F}\right)$.

Remark 2.3. The result indicates that, when both $R, J$ are locally polyhedral, the trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is a spiral. For the case $R, J$ being general partly smooth function, though we cannot prove, numerical evidence shows that the trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ could be either straight line or also a spiral.
$R$ or/and $J$ is smooth Now we consider the case that at least one function out of $R, J$ is smooth. For simplicity, consider that $R$ is smooth and $J$ remains non-smooth.

Proposition 2.4. For problem $(\mathcal{P})$ and $A D M M$ iteration (1.1), assume that conditions $(\mathcal{A} .1)-(\mathcal{A} .3)$ are true, then $\left(x_{k}, y_{k}, \psi_{k}\right)$ converges to a saddle point $\left(x^{\star}, y^{\star}, \psi^{\star}\right)$ of $\mathcal{L}(x, y ; \psi)$. Suppose $R$ is locally $C^{2}$ around $x^{\star}$, $J \in \operatorname{PSF}_{y^{\star}}\left(\mathcal{M}_{y^{\star}}^{J}\right)$ is partly smooth and condition (ND) holds for $J$, then Theorem 2.2(i) holds for all $k$ large enough. If moreover, $A$ is full rank square matrix, then all the eigenvalues of $M_{\mathrm{ADMM}}$ are real for $\gamma>$ $\left\|\left(A^{T} A\right)^{-\frac{1}{2}} \nabla^{2} R\left(x^{\star}\right)\left(A^{T} A\right)^{-\frac{1}{2}}\right\|$.

Remark 2.5. The real spectrum of $M$, numerical evidence shows that the eventual trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is a straight line, which is different from the case where both functions are non-smooth. If $o\left(\left\|v_{k-1}\right\|\right)$ is vanishing fast enough, we can also prove that $\theta_{k} \rightarrow 0$.

When $\gamma \leq\left\|\left(A^{T} A\right)^{-\frac{1}{2}} \nabla^{2} R\left(x^{\star}\right)\left(A^{T} A\right)^{-\frac{1}{2}}\right\|, M_{\text {ADMM }}$ will have complex eigenvalues, however the trajectory could be either spiral or straight line depending the leading eigenvalue. If both $R, J$ are smooth, $M$ will also have real spectrum under proper choice of $\gamma$.

In Figure 1 (a) and (c), we present two examples of the trajectory of ADMM. Subfigure (a) shows a spiral trajectory in $\mathbb{R}^{2}$ which is obtained from solving a polyhedral problem, while subfigure (c) is an eventual straight line trajectory in $\mathbb{R}^{3}$.

## 3 The failure of inertial acceleration

We use the LASSO problem as an example to demonstrate the effects of applying the inertial technique to ADMM, especially when it failures. One simple approach for combining inertial technique with ADMM is described below

$$
\begin{align*}
x_{k} & =\operatorname{argmin}_{x \in \mathbb{R}^{n}} R(x)+\frac{\gamma}{2}\left\|A x-\frac{1}{\gamma}\left(\bar{z}_{k-1}-2 \psi_{k-1}\right)\right\|^{2}, \\
z_{k} & =\psi_{k-1}+\gamma A x_{k}, \\
\bar{z}_{k} & =z_{k}+a_{k}\left(z_{k}-z_{k-1}\right),  \tag{3.1}\\
y_{k} & =\operatorname{argmin}_{y \in \mathbb{R}^{m}} J(y)+\frac{\gamma}{2}\left\|B y+\frac{1}{\gamma}\left(\bar{z}_{k}-\gamma b\right)\right\|^{2}, \\
\psi_{k} & =\bar{z}_{k}+\gamma\left(B y_{k}-b\right),
\end{align*}
$$

which considers only the momentum of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ without any stronger assumptions on $R, J$. The above scheme can reformulated as an instance of inertial Proximal Point Algorithm, guaranteed to be convergent for $a_{k}<\frac{1}{3}$ [3]; We refer to [32] or [25, Chapter 4.3] for more details.

The formulation of LASSO in the form of $(\mathcal{P})$ reads

$$
\begin{equation*}
\min _{x, y \in \mathbb{R}^{n}} \mu\|x\|_{1}+\frac{1}{2}\|K y-f\|^{2} \quad \text { such that } \quad x-y=0, \tag{3.2}
\end{equation*}
$$

where $K \in \mathbb{R}^{m \times n}, m<n$ is a random Gaussian matrix. Since $\frac{1}{2}\|K y-f\|^{2}$ is quadratic, owing to Proposition 2.4, the eventual trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is a straight line if $\gamma>\|K\|^{2}$, and a spiral for some $\gamma \leq\|K\|^{2}$. Therefore, we consider two different choices of $\gamma$ which are $\gamma=1$ and $\gamma=\|K\|^{2}$, and for each $\gamma$, four different choices of $a_{k}$ are considered

$$
a_{k} \equiv 0.3, \quad a_{k} \equiv 0.7 \quad \text { and } \quad a_{k}=\frac{k-1}{k+3} .
$$

The 3rd choice of $a_{k}$ corresponds to FISTA [14]. Numerical results are shown in Figure 1 (b) and (d),

- When $\gamma=1$, the inertial scheme works only for $a_{k} \equiv 0.3$, which is due to that fact that the trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is a spiral for $\gamma=1$. As a result, the direction $z_{k}-z_{k-1}$ is not pointing towards $z^{\star}$, hence unable to provide satisfactory acceleration.


Figure 1: Trajectory of sequence $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ and effects of inertial on ADMM. (a) Spiral trajectory of ADMM; (b) failure of inertial ADMM on spiral trajectory; (c) Eventual straight line trajectory; (d) success of inertial ADMM on straight line trajectory.

- When $\gamma=\|K\|^{2}$, all choices of $a_{k}$ work since $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ eventually forms a straight line. Among these four choices of $a_{k}, a_{k} \equiv 0.7$ is the fastest, while $a_{k}=\frac{k-1}{k+3}$ eventually is the slowest.
It should be noted that, though ADMM is faster for $\gamma=1$ than $\gamma=\|K\|^{2}$, our main focus here is to demonstrate how the trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ affects the outcome of inertial acceleration.

The above comparisons, particularly for $\gamma=1$ implies that the trajectory of the sequence $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is crucial for the acceleration outcome of the inertial scheme. Since the trajectories of ADMM depends on the properties of $R, J$ and choice of $\gamma$, this implies that the right scheme that can achieve uniform acceleration despite $R, J$ and $\gamma$ should be able to adapt itself to the trajectory of the method.

## $4 \quad \mathbf{A}^{3}$ DMM: adaptive acceleration for ADMM

The previous section shows the trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ eventually settles onto a regular path i.e. either straight line or spiral. In this section, we exploit this regularity to design adaptive acceleration for ADMM, which is called "A3 DMM"; See Algorithm 1.

The update of $\bar{z}_{k}$ in (3.1) can be viewed as a special case of the following extrapolation

$$
\begin{equation*}
\bar{z}_{k}=\mathcal{E}\left(z_{k}, z_{k-1}, \cdots, z_{k-q}\right) \tag{4.1}
\end{equation*}
$$

for the choice of $q=1$. The idea is: given $\left\{z_{k-j}\right\}_{j=0}^{q+1}$, define $v_{j} \stackrel{\text { def }}{=} z_{j}-z_{j-1}$ and predict the future iterates by considering how the past directions $v_{k-1}, \ldots, v_{k-q}$ approximate the latest direction $v_{k}$. In particular, define
$V_{k-1} \stackrel{\text { def }}{=}\left[v_{k-1}, \cdots, v_{k-q}\right] \in \mathbb{R}^{n \times q}$, and let $c_{k} \stackrel{\text { def }}{=} \operatorname{argmin}_{c \in \mathbb{R}^{q}}\left\|V_{k-1} c-v_{k}\right\|^{2}=\left\|\sum_{j=1}^{q} c_{j} v_{k-j}-v_{k}\right\|^{2}$. The idea is then that $V_{k} c_{k} \approx v_{k+1}$ and so, $\bar{z}_{k, 1} \stackrel{\text { def }}{=} z_{k}+V_{k} c \approx z_{k+1}$. By iterating this $s$ times, we obtain $\bar{z}_{k, s} \approx z_{k+s}$.

More precisely, given $c \in \mathbb{R}^{q}$, define the mapping $H$ by $H(c)=\left[\begin{array}{c|c}c_{1: q-1} & \mathrm{Id}_{q-1} \\ c_{q} & 0_{1, q-1}\end{array}\right] \in \mathbb{R}^{q \times q}$. Let $C_{k}=$ $H\left(c_{k}\right)$, note that $V_{k}=V_{k-1} C_{k}$. Define $\bar{V}_{k, 0} \stackrel{\text { def }}{=} V_{k}$ and for $s \geq 1$, define

$$
\bar{V}_{k, s} \stackrel{\text { def }}{=} \bar{V}_{k, s-1} C_{k} \stackrel{\text { def }}{=} V_{k} C_{k}^{s}
$$

where $C_{k}^{s}$ is the power of $C_{k}$. Let $(C)_{(:, 1)}$ be the first column of matrix $C$, then

$$
\begin{equation*}
\bar{z}_{k, s}=z_{k}+\sum_{i=1}^{s}\left(\bar{V}_{k, i}\right)_{(:, 1)}=z_{k}+\sum_{i=1}^{s} V_{k}\left(C_{k}^{i}\right)_{(:, 1)}=z_{k}+V_{k}\left(\sum_{i=1}^{s} C_{k}^{i}\right)_{(:, 1)} \tag{4.2}
\end{equation*}
$$

which is the desired trajectory following extrapolation scheme. Now define the extrapolation parameterised by $s, q$ as

$$
\mathcal{E}_{s, q}\left(z_{k}, \cdots, z_{k-q-1}\right) \stackrel{\text { def }}{=} V_{k}\left(\sum_{i=1}^{s} C_{k}^{i}\right)_{(:, 1)}
$$

we obtain the following trajectory following adaptive acceleration for ADMM.

```
Algorithm 1: \(\mathrm{A}^{3} \mathrm{DMM}\) : Adaptive Acceleration for ADMM
    Initial: Let \(s \geq 1, q \geq 1\) be integers and \(p=q+1\). Let \(\bar{z}_{0}=z_{0} \in \mathbb{R}^{n}\) and \(V_{0}=0 \in \mathbb{R}^{n \times q}\).
    Repeat:
        - For \(k \geq 1: \quad y_{k}=\operatorname{argmin}_{y \in \mathbb{R}^{m}} J(y)+\frac{\gamma}{2}\left\|B y+\frac{1}{\gamma}\left(\bar{z}_{k-1}-\gamma b\right)\right\|^{2}\),
                \(\psi_{k}=\bar{z}_{k-1}+\gamma\left(B y_{k}-b\right)\),
                    \(x_{k}=\operatorname{argmin}_{x \in \mathbb{R}^{n}} R(x)+\frac{\gamma}{2}\left\|A x-\frac{1}{\gamma}\left(\bar{z}_{k-1}-2 \psi_{k}\right)\right\|^{2}\),
                    \(z_{k}=\psi_{k}+\gamma A x_{k}\),
                    \(v_{k}=z_{k}-z_{k-1} \quad\) and \(\quad V_{k}=\left[v_{k}, V_{k-1}(:, 1: q-1)\right]\).
```

- If $\bmod (k, p)=0:$ Compute $C_{k}$ as described above, if $\rho\left(C_{k}\right)<1$ :

$$
\bar{z}_{k}=z_{k}+a_{k} \mathcal{E}_{s, q}\left(z_{k}, \cdots, z_{k-q-1}\right)
$$

- If $\bmod (k, p) \neq 0: \quad \bar{z}_{k}=z_{k}$.

Until: $\left\|v_{k}\right\| \leq$ tol.

## Remark 4.1.

- When $\bmod (k, p) \neq 0$, one can also consider $\bar{z}_{k}=z_{k}+a_{k}\left(z_{k}-z_{k-1}\right)$ with properly chosen $a_{k}$.
- $\mathrm{A}^{3} \mathrm{DMM}$ carries out $p$ standard ADMM iterations to set up the extrapolation step $\mathcal{E}_{s, q}$. As $\mathcal{E}_{s, q}$ contains the sum of the powers of $C_{k}$ which is guaranteed to be convergent when $\rho\left(C_{k}\right)<1$. Therefore, we only apply $\mathcal{E}_{s, q}$ when the spectral radius $\rho\left(C_{k}\right)<1$ is true. In this case, there is a closed form expression for $\mathcal{E}_{s, q}$ when $s=+\infty$; See Eq. (4.4).
- The purpose of adding $a_{k}$ in front of $\mathcal{E}_{s, q}\left(z_{k}, \cdots, z_{k-q-1}\right)$ is so that we can control the value of $a_{k}$ to ensure the convergence of the algorithm; See below the discussion.


### 4.1 Convergence of $\mathrm{A}^{3} \mathrm{DMM}$

To discuss the convergence of $\mathrm{A}^{3} \mathrm{DMM}$, we shall treat the algorithm as a perturbation of the original ADMM. If the perturbation error is absolutely summable, then we obtain the convergence of $\mathrm{A}^{3} \mathrm{DMM}$. More precisely, let $\varepsilon_{k} \in \mathbb{R}^{n}$ whose value takes

$$
\varepsilon_{k}=\left\{\begin{aligned}
& 0: \bmod (k, p) \neq 0 \text { or } \bmod (k, p)=0 \& \rho\left(C_{k}\right) \geq 1 \\
& a_{k} \mathcal{E}_{s, q}\left(z_{k}, \cdots, z_{k-q-1}\right): \bmod (k, p)=0 \& \rho\left(C_{k}\right)<1
\end{aligned}\right.
$$

Suppose the fixed-point formulation of ADMM can be written as $z_{k}=\mathcal{F}\left(z_{k-1}\right)$ for some $\mathcal{F}$ (see Section B of the appendix for details). Then Algorithm 1 can be written as

$$
\begin{equation*}
z_{k}=\mathcal{F}\left(z_{k-1}+\varepsilon_{k-1}\right) . \tag{4.3}
\end{equation*}
$$

Owing to (4.3), we can obtain the following convergence for Algorithm 1 which is based on the classic convergence result of inexact Krasnosel'skiǐ-Mann fixed-point iteration [5, Proposition 5.34].

Proposition 4.2. For problem $(\mathcal{P})$ and Algorithm 1, suppose that the conditions $(\mathcal{A} .1)-(\mathcal{A} .3)$ are true. If moreover, $\sum_{k}\left\|\varepsilon_{k}\right\|<+\infty, z_{k} \rightarrow z^{\star} \in \operatorname{fix}(\mathcal{F}) \stackrel{\text { def }}{=}\left\{z \in \mathbb{R}^{p}: z=\mathcal{F}(z)\right\}$ and $\left(x_{k}, y_{k}, \psi_{k}\right)$ converges to $\left(x^{\star}, y^{\star}, \psi^{\star}\right)$ which is a saddle point of $\mathcal{L}(x, y ; \psi)$.

On-line updating rule The summability condition $\sum_{k}\left\|\varepsilon_{k}\right\|<+\infty$ in general cannot be guaranteed. However, it can be enforced by a simple online updating rule. Let $a \in[0,1]$ and $b, \delta>0$, then $a_{k}$ can be determined by $a_{k}=\min \left\{a, b /\left(k^{1+\delta}\left\|z_{k}-z_{k-1}\right\|\right)\right\}$.

Inexact $\mathbf{A}^{3} \mathbf{D M M}$ Observe that in $\mathrm{A}^{3} \mathrm{DMM}$, when $A, B$ are non-trivial, in general there are no closed form solutions for $x_{k}$ and $y_{k}$. Take $x_{k}$ for example, suppose it is computed approximately, then in $z_{k}$ there will be another approximation error $\varepsilon_{k}^{\prime}$, and consequently

$$
z_{k}=\mathcal{F}\left(z_{k-1}+\varepsilon_{k-1}+\gamma \varepsilon_{k-1}^{\prime}\right)
$$

If there holds $\sum_{k}\left\|\varepsilon_{k-1}^{\prime}\right\|<+\infty$, Proposition 4.2 remains true for the above perturbation form.

### 4.2 Acceleration guarantee for $\mathrm{A}^{3} \mathrm{DMM}$

We have so far alluded to the idea that the extrapolated point $\bar{z}_{k, s}$ defined in (4.2) (which depends only on $\left.\left\{z_{k-j}\right\}_{j=0}^{q}\right)$ is an approximation to $z_{k+s}$. In this section, we make precise this statement.
Relationship to MPE and RRE We first show that $\bar{z}_{k, \infty}$ is (almost) equivalent to MPE. Recall that given a square matrix $C$, if its Neumann series is convergent, then there holds $(\operatorname{Id}-C)^{-1}=\sum_{i=0}^{+\infty} C^{i}$. Now for the summation of the power of $C_{k}$ in (4.2), when $s=+\infty$, we have

$$
\sum_{i=1}^{+\infty} C_{k}^{i}=C_{k} \sum_{i=0}^{+\infty} C_{k}^{i}=C_{k}\left(\operatorname{Id}-C_{k}\right)^{-1}=\left(\operatorname{Id}-C_{k}\right)^{-1}-\mathrm{Id} .
$$

Back to (4.2), then we get

$$
\begin{align*}
\bar{z}_{k, \infty} & \stackrel{\text { def }}{=} z_{k}+V_{k}\left(\left(\operatorname{Id}-C_{k}\right)^{-1}-\mathrm{Id}\right)_{(:, 1)}=z_{k}-v_{k}+V_{k}\left(\left(\operatorname{Id}-C_{k}\right)^{-1}\right)_{(:, 1)} \\
& =z_{k-1}+V_{k}\left(\left(\operatorname{Id}-C_{k}\right)^{-1}\right)_{(:, 1)}=\frac{1}{1-\sum_{i=1}^{s} c_{k, i}}\left(z_{k}-\sum_{j=1}^{q-1} c_{k, j} z_{k-j}\right), \tag{4.4}
\end{align*}
$$

which turns out to be MPE, with the slight difference of taking the weighted sum of $\left\{z_{j}\right\}_{j=k-q+1}^{k}$ as opposed to the weighted sum of $\left\{z_{j}\right\}_{j=k-q}^{k-1}$ (See appendix for more details of MPE). Note that if the coefficients $c$ is computed in the following way: $b \in \operatorname{argmin}_{a \in \mathbb{R}^{q+1}, \sum_{j} a_{j}=1}\left\|\sum_{j=0}^{q} a_{j} v_{k-j}\right\|$ and $b_{0} \neq 0$ and define $c_{j} \xlongequal{\text { def }}-b_{j} / b_{0}$ for $j=1, \ldots, q$. Then,

$$
\left(1-\sum_{i=1}^{q} c_{i}\right)^{-1}=\frac{b_{0}}{b_{0}+\sum_{j=1}^{q} b_{j}}=b_{0}
$$

and $\bar{z}_{k, \infty}=\sum_{j=0}^{q-1} b_{j} z_{k-j}$ is precisely the RRE update (again with the slight difference of summing over iterates shifted by one iteration).

Acceleration guarantee for $\mathbf{A}^{3} \mathbf{D M M}$ Let $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}^{n}$ and let $v_{k} \stackrel{\text { def }}{=} z_{k}-z_{k-1}$. Assume that $v_{k}=M v_{k-1}$ for some $M \in \mathbb{R}^{n \times n}$. Denote $\lambda(M)$ the spectrum of $M$. The following proposition provides control on the extrapolation error for $\bar{z}_{k, s}$ from (4.2).

Proposition 4.3. Define the coefficient fitting error by $\epsilon_{k} \stackrel{\text { def }}{=} \min _{c \in \mathbb{R}^{q}}\left\|V_{k-1} c-v_{k}\right\|$.
(i) For $s \in \mathbb{N}$, we have

$$
\begin{equation*}
\left\|\bar{z}_{k, s}-z^{\star}\right\| \leq\left\|z_{k+s}-z^{\star}\right\|+B_{s} \epsilon_{k} . \tag{4.5}
\end{equation*}
$$

where $B_{s} \stackrel{\text { def }}{=} \sum_{\ell=1}^{s}\left\|M^{\ell}\right\|\left|\sum_{i=0}^{s-\ell}\left(C_{k}^{i}\right)_{(1,1)}\right|$. If $\rho(M)<1$ and $\rho\left(C_{k}\right)<1$, then $\sum_{i} c_{k, i} \neq 1$ and $B_{s}$ is uniformly bounded in s. For $s=+\infty, B_{\infty} \stackrel{\text { def }}{=}\left|1-\sum_{i} c_{k, i}\right|^{-1} \sum_{\ell=1}^{\infty}\|M\|^{\ell}$
(ii) Suppose that $M$ is diagonalisable. Let $\left(\lambda_{j}\right)_{j}$ denote its distinct eigenvalues ordered such that $\left|\lambda_{j}\right| \geq\left|\lambda_{j+1}\right|$ and $\left|\lambda_{1}\right|=\rho(M)<1$. Suppose that $\left|\lambda_{q}\right|>\left|\lambda_{q+1}\right|$.

- Asymptotic bound (fixed $q$ and as $k \rightarrow+\infty)$ : $\epsilon_{k}=\mathcal{O}\left(\left|\lambda_{q+1}\right|^{k}\right)$.
- Nonasymptotic bound (fixed $q$ and $k$ ): Suppose that $\lambda(M)$ is real-valued and contained in the interval $[\alpha, \beta]$ with $-1<\alpha<\beta<1$. Then,

$$
\begin{equation*}
\frac{\epsilon_{k}}{1-\sum_{i} c_{k, i}} \leq K \beta^{k-q}\left(\frac{\sqrt{\eta}-1}{\sqrt{\eta}+1}\right)^{q} \tag{4.6}
\end{equation*}
$$

$$
\text { where } K \stackrel{\text { def }}{=} 2\left\|z_{0}-z^{\star}\right\|\left\|(\operatorname{Id}-M)^{\frac{1}{2}}\right\| \text { and } \eta=\frac{1-\alpha}{1-\beta} \text {. }
$$

## Remark 4.4.

- From Theorem 2.2(ii), when $R$ and $J$ are both polyhedral, we have a perfect local linearisation with the corresponding linearisation matrix being normal and hence, the conditions of Proposition 4.3 holds for all $k$ large enough. The first bound (i) shows that the extrapolated point $\bar{z}_{k, s}$ moves along the true trajectory as $s$ increases, up to the fitting error $\epsilon_{k}$. Although $\bar{z}_{k, \infty}$ is essentially an MPE update which is known to satisfy error bound (4.6) (see [38]), this proposition offers a further interpretation of these extrapolation methods in terms of following the "sequence trajectory", and combined with our local analysis of ADMM, provides justification of these methods for the acceleration of non-smooth optimisation problems.
- Proposition 4.3 (ii) shows that extrapolation improves the convergence rate from $\mathcal{O}\left(\left|\lambda_{1}\right|^{k}\right)$ to $\mathcal{O}\left(\left|\lambda_{q+1}\right|^{k}\right)$, and the nonasymptotic bound shows that the improvement of extrapolation is optimal in the sense of Nesterov [31]. Recalling the form of the eigenvalues of $M$ from Theorem 2.2, in the case of two nonsmooth polyhedral terms, we must have $\left|\lambda_{2 j-1}\right|=\left|\lambda_{2 j}\right|>\left|\lambda_{2 j+1}\right|$ for all $j \geq 1$. Hence, no acceleartion can be guaranteed or observed when $q=1$, while the choice of $q=2$ provides guaranteed acceleration.


## 5 Numerical experiments

Below we present numerical experiments on affine constrained minimisation (e.g. Basis Pursuit), LASSO, quadratic programming and image processing problems to demonstrate the performance of the proposed scheme. In the numerical comparison below, we mainly compare with the original ADMM and its inertial version (3.1) with fixed $a_{k} \equiv 0.3$. For the proposed $\mathrm{A}^{3} \mathrm{DMM}$, two settings are considered: $(q, p, s)=(6,7,100)$ and $(q, p, s)=(6,7,+\infty)$. The quantity we compare is $\left\|x_{k}-x^{\star}\right\|$.

### 5.1 Affine constrained minimisation

Consider the following constrained problem

$$
\begin{equation*}
\min _{x \in \mathbb{R}^{n}} R(x) \quad \text { such that } \quad K x=f \tag{5.1}
\end{equation*}
$$

Denote the set $\Omega \stackrel{\text { def }}{=}\left\{x \in \mathbb{R}^{n}: K x=f\right\}$, and $\iota_{\Omega}$ its indicator function. Then (5.1) can be written as

$$
\begin{equation*}
\min _{x, y \in \mathbb{R}^{n}} R(x)+\iota_{\Omega}(y) \quad \text { such that } \quad x-y=0, \tag{5.2}
\end{equation*}
$$

which is special case of $(\mathcal{P})$ with $A=\operatorname{Id}, B=-\mathrm{Id}$ and $b=0$. Here $K$ is generated from the standard Gaussian ensemble, and the following three choices of $R$ are considered:
$\ell_{1}$-norm $(m, n)=(512,2048)$, solution $x^{\star}$ is 128 -sparse;
$\ell_{1,2}$-norm $(m, n)=(512,2048)$, solution $x^{\star}$ has 32 non-zero blocks of size 4 ;
Nuclear norm $(m, n)=(1448,4096)$, solution $x^{\star}$ has rank of 4 .

The property of $\left\{\theta_{k}\right\}_{k \in \mathbb{N}}$ is shown in Figure 2 (a)-(c). Note that the indicator function $\iota_{\Omega}(y)$ in (5.2) is polyhedral since $\Omega$ is an affine subspace,

- As $\ell_{1}$-norm is polyhedral, we have in Figure 2(a) that $\theta_{k}$ is converging to a constant which complies with Theorem 2.2(ii).
- Since $\ell_{1,2}$-norm and nuclear norm are no longer polyhedral functions, we have that $\theta_{k}$ eventually oscillates in a range, meaning that the trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is an elliptical spiral.
Comparisons of the four schemes are shown below in Figure 2 (d)-(f):
- Since both functions in (5.2) are non-smooth, the eventual trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ for ADMM is spiral. Inertial ADMM fails to provide acceleration locally.
- $A^{3} D M M$ is faster than both $A D M M$ and inertial ADMM. For the two different settings of $\mathrm{A}^{3} \mathrm{DMM}$, their performances are very close.


Figure 2: Performance comparisons and $\left\{\theta_{k}\right\}_{k \in \mathbb{N}}$ of ADMM for affine constrained problem.

### 5.2 LASSO

We consider again the LASSO problem (3.2) with three datasets from LIBSVM ${ }^{1}$. The numerical experiments are provided below in Figure 3.

It can be observed that the proposed $A^{3} D M M$ is significantly faster than the other schemes, especially for $s=+\infty$. Between ADMM and inertial ADMM, different from the previous example, the inertial technique can provided consistent acceleration for all three examples.

[^1]

Figure 3: Performance comparisons for LASSO problem.

### 5.3 Quadratic programming

Consider the following quadratic optimisation problem

$$
\begin{align*}
\min _{x \in \mathbb{R}^{n}} & \frac{1}{2} x^{T} Q x+\langle q, x\rangle,  \tag{5.3}\\
\text { such that } & x_{i} \in\left[\ell_{i}, r_{i}\right], \quad i=1, \ldots, n .
\end{align*}
$$

Define the constraint set $\Omega=\left\{x \in \mathbb{R}^{n}: x_{i} \in\left[\ell_{i}, r_{i}\right], i=1, \ldots, n\right\}$, then (5.3) can be written as

$$
\min _{x, y \in \mathbb{R}^{n}} \quad \frac{1}{2} x^{T} Q x+\langle q, x\rangle+\iota_{\Omega}(y) \quad \text { such that } \quad x-y=0,
$$

which is special case of $(\mathcal{P})$ with $A=\operatorname{Id}, B=-\mathrm{Id}$ and $b=0$.
The angle $\theta_{k}$ of ADMM and the performances of the four schemes are provided in Figure (4), from which we observed that

- The angle $\theta_{k}$ is decreasing to 0 at the beginning and then starts to increasing for $k \geq 2 \times 10^{4}$. This is mainly due to the fact that for $k \geq 2 \times 10^{4}$, the effects of machine error is becoming increasingly larger.
- Consistent with the observations in Section 5, the proposed $A^{3}$ DMM schemes provides the best performance.


Figure 4: Performance comparisons and $\left\{\theta_{k}\right\}_{k \in \mathbb{N}}$ of ADMM for quadratic programming.

### 5.4 Total variation based image inpainting

Now we consider a total variation (TV) based image inpainting problem. Let $u \in \mathbb{R}^{n \times n}$ be an image and $\mathcal{S} \in \mathbb{R}^{n \times n}$ be a Bernoulli matrix, the observation of $u$ under $\mathcal{S}$ is $f=\mathcal{P}_{\mathcal{S}}(u)$. The TV based image inpainting
can be formulated as

$$
\begin{equation*}
\min _{x \in \mathbb{R}^{n \times n}}\|\nabla x\|_{1} \quad \text { such that } \quad \mathcal{P}_{\mathcal{S}}(x)=f . \tag{5.4}
\end{equation*}
$$

Define $\Omega \stackrel{\text { def }}{=}\left\{x \in \mathbb{R}^{n \times n}: \mathcal{P}_{\mathcal{S}}(x)=f\right\}$, then (5.4) becomes

$$
\begin{equation*}
\min _{x \in \mathbb{R}^{n \times n}}\|y\|_{1}+\iota_{\Omega}(x) \quad \text { such that } \quad \nabla x-y=0, \tag{5.5}
\end{equation*}
$$

which is special case of $(\mathcal{P})$ with $A=\nabla, B=-\operatorname{Id}$ and $b=0$. For the update of $x_{k}$, we have from (1.2) that

$$
x_{k}=\operatorname{argmin}_{x \in \mathbb{R}^{n \times n}} \iota_{\Omega}(x)+\frac{\gamma}{2}\left\|\nabla x-\frac{1}{\gamma}\left(\bar{z}_{k-1}-2 \psi_{k-1}\right)\right\|^{2},
$$

which does not admit closed form solution. In the implementation, finite-step FISTA is applied to roughly solve the above problem.

In the experiment, the cameraman image is used, and $50 \%$ of the pixels is removed randomly. The angle $\theta_{k}$ of ADMM and the comparisons of the four schemes are provided in Figure 5:

- Though both functions in (5.5) are polyhedral, since the subproblem of $x_{k}$ is solved approximately, the eventual angle actually is oscillating instead of being a constant.
- Inertial ADMM again is slower than the original ADMM as the trajectory of ADMM is a spiral.
- For the two $\mathrm{A}^{3} \mathrm{DMM}$ schemes, their performances are close as previous examples.
- For PSNR the image quality assessment, Figure 5(c) implies that $\mathrm{A}^{3} \mathrm{DMM}$ is also the best.


Figure 5: Property of $\left\{\theta_{k}\right\}_{k \in \mathbb{N}}$, performance comparison and image quality of ADMM for TV based image inpainting.

We also compare the visual quality of the images obtained by the four schemes for the 8 'th iteration, which is shown below in Figure 6. Since we choose $(q, p)=(6,7)$, for $k=8$ both $\mathrm{A}^{3}$ DMM applies only one step adaptive acceleration step, and the image quality (2nd row of Figure 6 ) is much better than the 1 st row of ADMM and inertial ADMM.

## 6 Conclusions

In this article, by analysing the trajectory of the fixed point sequences associated to ADMM and extrapolating along the trajectory, we provide an alternative derivation of these methods. Furthermore, our local linear analysis allows for the application of previous results on extrapolation methods, and hence provides guaranteed (local) acceleration. Extension of the proposed acceleration framework to general first-order methods is ongoing.

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Figure 6: Comparison of image quality at the $8^{\prime}$ th iteration of ADMM , inertial ADMM and two $\mathrm{A}^{3} \mathrm{DMM}$ schemes.
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## Appendix

The organisation of the appendix is as follows: the proofs of the main results of the paper are contained in Sections AC, where in Section A some preliminary result on angles between subspaces and Riemannian geometry are provide, in Section B the proofs for the trajectory of ADMM are provided, and lastly in in Section C we provide proofs on A ${ }^{3}$ DMM.

## A Preliminaries

## A. 1 Polynomial extrapolation

Minimal polynomial extrapolation (MPE) [13]: Given $\left\{z_{k-j}\right\}_{j=0}^{q+1}$, let $\left\{v_{k-j}\right\}_{j=0}^{q}$ be the difference vectors, where $v_{j} \stackrel{\text { def }}{=}$ $z_{j}-z_{j-1}$. Define $V_{k}=\left[\begin{array}{lll}v_{k} & \cdots & v_{k-q}\end{array}\right]$.

1. Let $\left\{c_{j}\right\}_{j=1}^{q} \in \operatorname{argmin}_{c \in \mathbb{R}^{q}}\left\|V_{k-1} c-v_{k}\right\|$, define $c_{0} \stackrel{\text { def }}{=} 1$ and $\gamma_{i}=c_{i} / \sum_{i=0}^{q} c_{i}$ for $i=0, \ldots, q$.
2. The extrapolated point is then defined to be $\bar{z}_{k} \stackrel{\text { def }}{=} \sum_{i=0}^{q} \gamma_{i} z_{k-i-1}$.

Reduced rank extrapolation (RRE) [18, 29] is obtained by replacing the first step by

$$
\left\{\gamma_{j}\right\}_{j=0}^{q} \in \operatorname{argmin}_{\gamma \in \mathbb{R}^{q+1}}\left\|V_{k} \gamma\right\| \text { subject to } \sum_{i} \gamma_{i}=1
$$

The motivation for the use of such methods for the acceleration of fixed point sequences $x_{k+1}=\mathcal{F}\left(z_{k}\right)$ come from considering the spectral properties of the linearisation around the limit point. In particular, if $z^{\star}$ is the limit point and $z_{k+1}-z^{\star}=T\left(z_{k}-z^{\star}\right)$ where $T \in \mathbb{R}^{d \times d}$ and $q$ is the order of the minimal polynomial of $T$ with respect to $z_{k-q-1}-z^{\star}$ (i.e. $q$ is the monic polynomial of least degree such that $P(T)\left(z_{k-q-1}-z^{\star}\right)=0$ ), then one can show that $\bar{z}_{k}=z^{\star}$. We refer to $[39,40,38]$ for details on these methods and their acceleratioon guarantees.

## A. 2 Angle between subspaces

Let $T_{1}, T_{2}$ be two subspaces, and without the loss of generality, assume

$$
1 \leq p \stackrel{\text { def }}{=} \operatorname{dim}\left(T_{1}\right) \leq q \stackrel{\text { def }}{=} \operatorname{dim}\left(T_{2}\right) \leq n-1
$$

Definition A. 1 (Principal angles). The principal angles $\theta_{k} \in\left[0, \frac{\pi}{2}\right], k=1, \ldots, p$ between subspaces $T_{1}$ and $T_{2}$ are defined by, with $u_{0}=v_{0} \stackrel{\text { def }}{=} 0$, and

$$
\begin{aligned}
\cos \left(\theta_{k}\right) \stackrel{\text { def }}{=}\left\langle u_{k}, v_{k}\right\rangle=\max \langle u, v\rangle \text { s.t. } & u \in T_{1}, v \in T_{2},\|u\|=1,\|v\|=1 \\
& \left\langle u, u_{i}\right\rangle=\left\langle v, v_{i}\right\rangle=0, i=0, \cdots, k-1 .
\end{aligned}
$$

The principal angles $\theta_{k}$ are unique and satisfy $0 \leq \theta_{1} \leq \theta_{2} \leq \cdots \leq \theta_{p} \leq \pi / 2$.
Definition A. 2 (Friedrichs angle). The Friedrichs angle $\left.\left.\theta_{F} \in\right] 0, \frac{\pi}{2}\right]$ between $T_{1}$ and $T_{2}$ is

$$
\cos \left(\theta_{F}\left(T_{1}, T_{2}\right)\right) \stackrel{\text { def }}{=} \max \langle u, v\rangle \text { s.t. } u \in T_{1} \cap\left(T_{1} \cap T_{2}\right)^{\perp},\|u\|=1, v \in T_{2} \cap\left(T_{1} \cap T_{2}\right)^{\perp},\|v\|=1
$$

The following lemma shows the relation between the Friedrichs and principal angles, whose proof can be found in [6, Proposition 3.3].

Lemma A. 3 (Principal angles and Friedrichs angle). The Friedrichs angle is exactly $\theta_{d+1}$ where $d \stackrel{\text { def }}{=} \operatorname{dim}\left(T_{1} \cap T_{2}\right)$. Moreover, $\theta_{F}\left(T_{1}, T_{2}\right)>0$.

## A. 3 Riemannian Geometry

Let $\mathcal{M}$ be a $C^{2}$-smooth embedded submanifold of $\mathbb{R}^{n}$ around a point $x$. With some abuse of terminology, we shall state $C^{2}$-manifold instead of $C^{2}$-smooth embedded submanifold of $\mathbb{R}^{n}$. The natural embedding of a submanifold $\mathcal{M}$ into $\mathbb{R}^{n}$ permits to define a Riemannian structure and to introduce geodesics on $\mathcal{M}$, and we simply say $\mathcal{M}$ is a Riemannian manifold. We denote respectively $\mathcal{T}_{\mathcal{M}}(x)$ and $\mathcal{N}_{\mathcal{M}}(x)$ the tangent and normal space of $\mathcal{M}$ at point near $x$ in $\mathcal{M}$.
Exponential map Geodesics generalize the concept of straight lines in $\mathbb{R}^{n}$, preserving the zero acceleration characteristic, to manifolds. Roughly speaking, a geodesic is locally the shortest path between two points on $\mathcal{M}$. We denote by $\mathfrak{g}(t ; x, h)$ the value at $t \in \mathbb{R}$ of the geodesic starting at $\mathfrak{g}(0 ; x, h)=x \in \mathcal{M}$ with velocity $\dot{\mathfrak{g}}(t ; x, h)=\frac{d \mathfrak{g}}{d t}(t ; x, h)=$ $h \in \mathcal{T}_{\mathcal{M}}(x)$ (which is uniquely defined). For every $h \in \mathcal{T}_{\mathcal{M}}(x)$, there exists an interval $I$ around 0 and a unique geodesic $\mathfrak{g}(t ; x, h): I \rightarrow \mathcal{M}$ such that $\mathfrak{g}(0 ; x, h)=x$ and $\dot{\mathfrak{g}}(0 ; x, h)=h$. The mapping

$$
\operatorname{Exp}_{x}: \mathcal{T}_{\mathcal{M}}(x) \rightarrow \mathcal{M}, h \mapsto \operatorname{Exp}_{x}(h)=\mathfrak{g}(1 ; x, h)
$$

is called Exponential map. Given $x, x^{\prime} \in \mathcal{M}$, the direction $h \in \mathcal{T}_{\mathcal{M}}(x)$ we are interested in is such that

$$
\operatorname{Exp}_{x}(h)=x^{\prime}=\mathfrak{g}(1 ; x, h)
$$

Parallel translation Given two points $x, x^{\prime} \in \mathcal{M}$, let $\mathcal{T}_{\mathcal{M}}(x), \mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)$ be their corresponding tangent spaces. Define

$$
\tau: \mathcal{T}_{\mathcal{M}}(x) \rightarrow \mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)
$$

the parallel translation along the unique geodesic joining $x$ to $x^{\prime}$, which is isomorphism and isometry w.r.t. the Riemannian metric.

Riemannian gradient and Hessian For a vector $v \in \mathcal{N}_{\mathcal{M}}(x)$, the Weingarten map of $\mathcal{M}$ at $x$ is the operator $\mathfrak{W}_{x}(\cdot, v): \mathcal{T}_{\mathcal{M}}(x) \rightarrow \mathcal{T}_{\mathcal{M}}(x)$ defined by

$$
\mathfrak{W}_{x}(\cdot, v)=-\mathcal{P}_{\mathcal{T}_{\mathcal{M}}(x)} \mathrm{d} V[h],
$$

where $V$ is any local extension of $v$ to a normal vector field on $\mathcal{M}$. The definition is independent of the choice of the extension $V$, and $\mathfrak{W}_{x}(\cdot, v)$ is a symmetric linear operator which is closely tied to the second fundamental form of $\mathcal{M}$, see [15, Proposition II.2.1].

Let $G$ be a real-valued function which is $C^{2}$ along the $\mathcal{M}$ around $x$. The covariant gradient of $G$ at $x^{\prime} \in \mathcal{M}$ is the vector $\nabla_{\mathcal{M}} G\left(x^{\prime}\right) \in \mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)$ defined by

$$
\left\langle\nabla_{\mathcal{M}} G\left(x^{\prime}\right), h\right\rangle=\left.\frac{d}{d t} G\left(\mathcal{P}_{\mathcal{M}}\left(x^{\prime}+t h\right)\right)\right|_{t=0}, \forall h \in \mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)
$$

where $\mathcal{P}_{\mathcal{M}}$ is the projection operator onto $\mathcal{M}$. The covariant Hessian of $G$ at $x^{\prime}$ is the symmetric linear mapping $\nabla_{\mathcal{M}}^{2} G\left(x^{\prime}\right)$ from $\mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)$ to itself which is defined as

$$
\begin{equation*}
\left\langle\nabla_{\mathcal{M}}^{2} G\left(x^{\prime}\right) h, h\right\rangle=\left.\frac{d^{2}}{d t^{2}} G\left(\mathcal{P}_{\mathcal{M}}\left(x^{\prime}+t h\right)\right)\right|_{t=0}, \forall h \in \mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right) \tag{A.1}
\end{equation*}
$$

This definition agrees with the usual definition using geodesics or connections [30]. Now assume that $\mathcal{M}$ is a Riemannian embedded submanifold of $\mathbb{R}^{n}$, and that a function $G$ has a $C^{2}$-smooth restriction on $\mathcal{M}$. This can be characterized by the existence of a $C^{2}$-smooth extension (representative) of $G$, i.e. a $C^{2}$-smooth function $\widetilde{G}$ on $\mathbb{R}^{n}$ such that $\widetilde{G}$ agrees with $G$ on $\mathcal{M}$. Thus, the Riemannian gradient $\nabla_{\mathcal{M}} G\left(x^{\prime}\right)$ is also given by

$$
\begin{equation*}
\nabla_{\mathcal{M}} G\left(x^{\prime}\right)=\mathcal{P}_{\mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)} \nabla \widetilde{G}\left(x^{\prime}\right) \tag{A.2}
\end{equation*}
$$

and $\forall h \in \mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)$, the Riemannian Hessian reads

$$
\begin{align*}
& \nabla_{\mathcal{M}}^{2} G\left(x^{\prime}\right) h=\mathcal{P}_{\mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)} \mathrm{d}\left(\nabla_{\mathcal{M}} G\right)\left(x^{\prime}\right)[h]=\mathcal{P}_{\mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)} \mathrm{d}\left(x^{\prime} \mapsto \mathcal{P}_{\mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)} \nabla_{\mathcal{M}} \widetilde{G}\right)[h] \\
&=\mathcal{P}_{\mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)} \nabla^{2} \widetilde{G}\left(x^{\prime}\right) h+\mathfrak{W}_{x^{\prime}}\left(h, \mathcal{P}_{\mathcal{N} \mathcal{M}}\left(x^{\prime}\right)\right.  \tag{A.3}\\
& \nabla \\
&\left.G\left(x^{\prime}\right)\right)
\end{align*}
$$

where the last equality comes from [1, Theorem 1]. When $\mathcal{M}$ is an affine or linear subspace of $\mathbb{R}^{n}$, then obviously $\mathcal{M}=x+\mathcal{T}_{\mathcal{M}}(x)$, and $\mathfrak{W}_{x^{\prime}}\left(h, \mathcal{P}_{\mathcal{N}_{\mathcal{M}}\left(x^{\prime}\right)} \nabla \widetilde{G}\left(x^{\prime}\right)\right)=0$, hence (A.3) reduces to

$$
\nabla_{\mathcal{M}}^{2} G\left(x^{\prime}\right)=\mathcal{P}_{\mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)} \nabla^{2} \widetilde{G}\left(x^{\prime}\right) \mathcal{P}_{\mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right)}
$$

See [23, 15] for more materials on differential and Riemannian manifolds.

## A. 4 Preparatory lemmas

The following lemmas characterise the parallel translation and the Riemannian Hessian of nearby points in $\mathcal{M}$.
Lemma A. 4 ([26, Lemma 5.1]). Let $\mathcal{M}$ be a $C^{2}$-smooth manifold around $x$. Then for any $x^{\prime} \in \mathcal{M} \cap \mathcal{N}$, where $\mathcal{N}$ is a neighbourhood of $x$, the projection operator $\mathcal{P}_{\mathcal{M}}\left(x^{\prime}\right)$ is uniquely valued and $C^{1}$ around $x$, and thus

$$
x^{\prime}-x=\mathcal{P}_{\mathcal{T}_{\mathcal{M}}(x)}\left(x^{\prime}-x\right)+o\left(\left\|x^{\prime}-x\right\|\right)
$$

If moreover $\mathcal{M}=x+\mathcal{T}_{\mathcal{M}}(x)$ is an affine subspace, then $x^{\prime}-x=\mathcal{P}_{\mathcal{T}_{\mathcal{M}}(x)}\left(x^{\prime}-x\right)$.
Lemma A. 5 ([27, Lemma B.1]). Let $x \in \mathcal{M}$, and $x_{k}$ a sequence converging to $x$ in $\mathcal{M}$. Denote $\tau_{k}: \mathcal{T}_{\mathcal{M}}\left(x_{k}\right) \rightarrow \mathcal{T}_{\mathcal{M}}(x)$ be the parallel translation along the unique geodesic joining $x$ to $x_{k}$. Then, for any bounded vector $u \in \mathbb{R}^{n}$, we have

$$
\left(\tau_{k} \mathcal{P}_{\mathcal{T}_{\mathcal{M}}\left(x_{k}\right)}-\mathcal{P}_{\mathcal{T}_{\mathcal{M}}(x)}\right) u=o(\|u\|)
$$

The Riemannian gradient and Hessian of partly smooth functions are covered by the lemma below.
Lemma A. 6 ([27, Lemma B.2]). Let $x, x^{\prime}$ be two close points in $\mathcal{M}$, denote $\tau: \mathcal{T}_{\mathcal{M}}\left(x^{\prime}\right) \rightarrow \mathcal{T}_{\mathcal{M}}(x)$ the parallel translation along the unique geodesic joining $x$ to $x^{\prime}$. The Riemannian Taylor expansion of $R \in C^{2}(\mathcal{M})$ around $x$ reads,

$$
\tau \nabla_{\mathcal{M}} R\left(x^{\prime}\right)=\nabla_{\mathcal{M}} R(x)+\nabla_{\mathcal{M}}^{2} R(x) \mathcal{P}_{\mathcal{T}_{\mathcal{M}}(x)}\left(x^{\prime}-x\right)+o\left(\left\|x^{\prime}-x\right\|\right)
$$

Lemma A. 7 (Riemannian gradient and Hessian). If $R \in \operatorname{PSF}_{x}\left(\mathcal{M}_{x}\right)$, then for any point $x^{\prime} \in \mathcal{M}_{x}$ near $x$

$$
\nabla_{\mathcal{M}_{x}} R\left(x^{\prime}\right)=\mathcal{P}_{T_{x^{\prime}}}\left(\partial R\left(x^{\prime}\right)\right)
$$

and this does not depend on the smooth representation of $R$ on $\mathcal{M}_{x}$. In turn, for all $h \in T_{x^{\prime}}$, let $\widetilde{R}$ be a smooth representative of $R$ on $\mathcal{M}_{x}$,

$$
\nabla_{\mathcal{M}_{x}}^{2} R\left(x^{\prime}\right) h=\mathcal{P}_{T_{x^{\prime}}} \nabla^{2} \widetilde{R}\left(x^{\prime}\right) h+\mathfrak{W}_{x^{\prime}}\left(h, \mathcal{P}_{T_{x^{\prime}}^{\perp}} \nabla \widetilde{R}\left(x^{\prime}\right)\right)
$$

where $\mathfrak{W}_{x}(\cdot, \cdot): T_{x} \times T_{x}^{\perp} \rightarrow T_{x}$ is the Weingarten map of $\mathcal{M}_{x}$ at $x$.

## A. 5 Linearisation of proximal mapping

In this part, we present one fundamental result led by partial smoothness, the linearisation of proximal mapping. We first discuss the property of the Riemannian Hessian of a partly smooth function. Let $R \in \Gamma_{0}\left(\mathbb{R}^{n}\right)$ be partly smooth at $\bar{x}$ relative to $\mathcal{M}_{\bar{x}}$ and $\bar{u} \in \partial R(\bar{x})$, define the following smooth perturbation of $R$

$$
\bar{R}(x) \stackrel{\text { def }}{=} R(x)-\langle x, \bar{u}\rangle
$$

whose Riemannian Hessian at $\bar{x}$ reads $H_{\bar{R}} \stackrel{\text { def }}{=} \mathcal{P}_{T_{\bar{x}}} \nabla_{\mathcal{M}_{\bar{x}}}^{2} \bar{R}(\bar{x}) \mathcal{P}_{T_{\bar{x}}}$.
Lemma A. 8 ([27, Lemma 4.2]). Let $R \in \Gamma_{0}\left(\mathbb{R}^{n}\right)$ be partly smooth at $\bar{x}$ relative to $\mathcal{M}_{\bar{x}}$, then $H_{\bar{R}}$ is symmetric positive semi-definite if either of the following is true:

- $\bar{u} \in \operatorname{ri}(\partial R(\bar{x}))$ is non-denegerate.
- $\mathcal{M}_{\bar{x}}$ is an affine subspace.

In turn, $\operatorname{Id}+H_{\bar{R}}$ is invertible and $M_{\bar{R}} \stackrel{\text { def }}{=}\left(\operatorname{Id}+H_{\bar{R}}\right)^{-1}$ is symmetric positive definite with all eigenvalues in $\left.] 0,1\right]$.
One consequence of Lemma A. 8 is that, we can linearise the generalised proximal mapping. For the sake of generality, let $\gamma>0, R \in \Gamma_{0}\left(\mathbb{R}^{n}\right)$ and $A \in \mathbb{R}^{p \times n}$, define the following generalised proximal mapping

$$
\operatorname{prox}_{\gamma R}^{A}(\cdot) \stackrel{\text { def }}{=} \operatorname{argmin}_{x \in \mathbb{R}^{n}} \gamma R(x)+\frac{1}{2}\|A x-\cdot\|^{2}
$$

Clearly, prox $\gamma_{\gamma R}^{A}$ is a single-valued mapping when $A$ has full column rank. Define $A_{T_{\bar{x}}}=A \circ \mathcal{P}_{T_{\bar{x}}}$, which has full column rank owing to $A$. Hence $A_{T_{\bar{x}}}^{T} A_{T_{\bar{x}}}$ is invertible. Denote

$$
M_{\bar{R}}=A_{T_{\bar{x}}}\left(\operatorname{Id}+\left(A_{T_{\bar{x}}}^{T} A_{T_{\bar{x}}}\right)^{-1} H_{\bar{R}}\right)^{-1}\left(A_{T_{\bar{x}}}^{T} A_{T_{\bar{x}}}\right)^{-1} A_{T_{\bar{x}}}^{T}
$$

Lemma A.9. Let function $R \in \Gamma_{0}\left(\mathbb{R}^{n}\right)$ be partly smooth at the point $\bar{x}$ relative to the manifold $\mathcal{M}_{\bar{x}}$ and $\bar{u} \in \operatorname{ri}(\partial R(\bar{x}))$. Suppose that there exists $\gamma>0$, full column rank $A \in \mathbb{R}^{p \times n}$ and $\bar{w} \in \mathbb{R}^{p}$ such that $\bar{x}=\operatorname{prox}_{\gamma R}^{A}(\bar{w})$ and $\bar{u}=-A^{T}(A \bar{x}-$ $\bar{w}) / \gamma$. Let $\left\{w_{k}\right\}_{k \in \mathbb{N}}$ be a sequence such that $w_{k} \rightarrow \bar{w}$ and $x_{k}=\operatorname{prox}_{\gamma R}^{A}\left(w_{k}\right) \rightarrow \bar{x}$, then for all $k$ large enough, there holds

$$
\begin{equation*}
A_{T_{\bar{x}}}\left(x_{k}-x_{k-1}\right)=M_{\bar{R}}\left(w_{k}-w_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right) . \tag{A.4}
\end{equation*}
$$

Remark A.10. When $A=\mathrm{Id}$, then $\operatorname{prox}_{\gamma R}^{A}$ reduces to the standard proximal mapping, and (A.4) simplifies to

$$
x_{k}-x_{k-1}=\mathcal{P}_{T_{\bar{x}}}\left(\operatorname{Id}+H_{\bar{R}}\right)^{-1} \mathcal{P}_{T_{\bar{x}}}\left(w_{k}-w_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right)
$$

In [25] and references therein, to study the local linear convergence of first-order methods, linearisation with respect to the limiting points is provided, that is

$$
x_{k}-\bar{x}=\mathcal{P}_{T_{\bar{x}}}\left(\operatorname{Id}+H_{\bar{R}}\right)^{-1} \mathcal{P}_{T_{\bar{x}}}\left(w_{k}-\bar{w}\right)+o\left(\left\|w_{k}-\bar{w}\right\|\right) .
$$

Proof. Since $R$ is proper convex and lower semi-continuous, we have $R\left(x_{k}\right) \rightarrow R(\bar{x})$ and $\partial R\left(x_{k}\right) \ni u_{k}=-A^{T}\left(A x_{k}-\right.$ $\left.w_{k}\right) / \gamma \rightarrow \bar{u} \in \operatorname{ri}(\partial R(\bar{x}))$, and we have $x_{k} \in \mathcal{M}_{\bar{x}}$ owing to [21, Theorem 5.3] and $u_{k} \in \operatorname{ri}\left(\partial R\left(x_{k}\right)\right)$ owing to [41] for all $k$ large enough.

Denote $T_{x_{k}}, T_{x_{k-1}}$ the tangent spaces of $\mathcal{M}_{\bar{x}}$ at $x_{k}$ and $x_{k-1}$. Denote $\tau_{k}: T_{x_{k}} \rightarrow T_{x_{k-1}}$ the parallel translation along the unique geodesic on $\mathcal{M}_{\bar{x}}$ joining $x_{k}$ to $x_{k-1}$.

From the definition of $x_{k}$, we get

$$
u_{k} \stackrel{\text { def }}{=}-A^{T}\left(A x_{k}-w_{k}\right) \in \gamma \partial R\left(x_{k}\right) \quad \text { and } \quad u_{k-1} \stackrel{\text { def }}{=}-A^{T}\left(A x_{k-1}-w_{k-1}\right) \in \gamma \partial R\left(x_{k-1}\right)
$$

Projecting on the corresponding tangent spaces, applying Lemma A. 7 and the parallel translation $\tau_{k}$ leads to

$$
\begin{aligned}
\gamma \tau_{k} \nabla_{\mathcal{M}_{\bar{x}}} R\left(x_{k}\right) & =\tau_{k} \mathcal{P}_{T_{x_{k}}^{R}}\left(u_{k}\right)=\mathcal{P}_{T_{x_{k-1}}^{R}}\left(u_{k}\right)+\left(\tau_{k} \mathcal{P}_{T_{x_{k}}^{J}}-\mathcal{P}_{T_{x_{k-1}}^{R}}\right)\left(u_{k}\right), \\
\gamma \nabla_{\mathcal{M}_{\bar{x}}} R\left(x_{k-1}\right) & =\mathcal{P}_{T_{x_{k-1}}^{R}}\left(u_{k-1}\right)
\end{aligned}
$$

The difference of the above two equalities leads to

$$
\begin{align*}
& \gamma \tau_{k} \nabla_{\mathcal{M}_{\bar{x}}} R\left(x_{k}\right)-\gamma \nabla_{\mathcal{M}_{\bar{x}}} R\left(x_{k-1}\right)-\left(\tau_{k} \mathcal{P}_{T_{x_{k}}^{J}}-\mathcal{P}_{T_{x_{k-1}}^{R}}\right)\left(u_{k-1}\right) \\
& =\mathcal{P}_{T_{x_{k-1}}^{R}}\left(u_{k}-u_{k-1}\right)+\left(\tau_{k} \mathcal{P}_{T_{x_{k}}^{J}}-\mathcal{P}_{T_{x_{k-1}}^{R}}\right)\left(u_{k}-u_{k-1}\right) \tag{A.5}
\end{align*}
$$

From the monotonicity from subdifferential, i.e. $\left\langle u_{k}-u_{k-1}, x_{k}-x_{k-1}\right\rangle \geq 0$, we get

$$
\left\langle A^{T} A\left(x_{k}-x_{k-1}\right), x_{k}-x_{k-1}\right\rangle \leq\left\langle A^{T}\left(w_{k}-w_{k-1}\right), x_{k}-x_{k-1}\right\rangle \leq\|A\|\left\|w_{k}-w_{k-1}\right\|\left\|x_{k}-x_{k-1}\right\|
$$

Since $A$ has full column rank, then $A^{T} A$ is symmetric positive definite, and there exists $\kappa>0$ such that $\kappa\left\|x_{k}-x_{k-1}\right\|^{2} \leq$ $\left\langle A^{T} A\left(x_{k}-x_{k-1}\right), x_{k}-x_{k-1}\right\rangle$. Back to the above inequality, we get $\left\|x_{k}-x_{k-1}\right\| \leq \frac{\|A\|}{\kappa}\left\|w_{k}-w_{k-1}\right\|$. Therefore for $\left\|u_{k}-u_{k-1}\right\|$, we get

$$
\begin{aligned}
\left\|u_{k}-u_{k-1}\right\|=\left\|A^{T}\left(A x_{k}-w_{k}\right)-A^{T}\left(A x_{k-1}-w_{k-1}\right)\right\| & \leq\|A\|^{2}\left\|x_{k}-x_{k-1}\right\|+\|A\|\left\|w_{k}-w_{k-1}\right\| \\
& \leq\left(\frac{\|A\|^{3}}{\kappa}+\|A\|\right)\left\|w_{k}-w_{k-1}\right\|
\end{aligned}
$$

As a result, owing to Lemma A.5, we have for the term $\left(\tau_{k} \mathcal{P}_{T_{x_{k}}^{J}}-\mathcal{P}_{T_{x_{k-1}}^{R}}\right)\left(u_{k}-u_{k-1}\right)$ in (A.5)

$$
\left(\tau_{k} \mathcal{P}_{T_{x_{k}}^{J}}-\mathcal{P}_{T_{x_{k-1}}^{R}}\right)\left(u_{k}-u_{k-1}\right)=o\left(\left\|w_{k}-w_{k-1}\right\|\right)
$$

Define $\bar{R}_{k-1}(x) \stackrel{\text { def }}{=} \gamma R(x)-\left\langle x, u_{k-1}\right\rangle$ and $H_{\bar{R}, k-1} \stackrel{\text { def }}{=} \mathcal{P}_{T_{x_{k-1}}} \nabla_{\mathcal{M}_{\bar{x}}}^{2} \bar{R}\left(x_{k-1}\right) \mathcal{P}_{T_{x_{k-1}}}$, then with Lemma A. 6 the Riemannian Taylor expansion, we have for the first line of (A.5)

$$
\begin{align*}
& \gamma \tau_{k} \nabla_{\mathcal{M}_{\bar{x}}} R\left(x_{k}\right)-\gamma \nabla_{\mathcal{M}_{\bar{x}}} R\left(x_{k-1}\right)-\left(\tau_{k} \mathcal{P}_{T_{x_{k}}}-\mathcal{P}_{T_{x_{k-1}}}\right)\left(u_{k-1}\right) \\
& =\tau_{k}\left(\gamma \nabla_{\mathcal{M}_{\bar{x}}} R\left(x_{k}\right)-\mathcal{P}_{T_{x_{k}}^{J}}\left(u_{k-1}\right)\right)-\left(\gamma \nabla_{\mathcal{M}_{\bar{x}}} R\left(x_{k-1}\right)-\mathcal{P}_{T_{x_{k-1}}}\left(u_{k-1}\right)\right) \\
& =\tau_{k} \nabla_{\mathcal{M}_{\bar{x}}} \bar{R}_{k-1}\left(x_{k}\right)-\nabla_{\mathcal{M}_{\bar{x}}} \bar{R}_{k-1}\left(x_{k-1}\right)  \tag{A.6}\\
& =H_{\bar{R}, k-1}\left(x_{k}-x_{k-1}\right)+o\left(\left\|x_{k}-x_{k-1}\right\|\right) \\
& =H_{\bar{R}, k-1}\left(x_{k}-x_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right) .
\end{align*}
$$

Back to (A.5), we get

$$
\begin{equation*}
H_{\bar{R}, k-1}\left(x_{k}-x_{k-1}\right)=\mathcal{P}_{T_{x_{k-1}}^{R}}\left(u_{k}-u_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right) \tag{A.7}
\end{equation*}
$$

Define $\bar{R}(x) \stackrel{\text { def }}{=} \gamma R(x)-\langle x, \bar{u}\rangle$ and $H_{\bar{R}}=\mathcal{P}_{T_{\bar{x}}} \nabla_{\mathcal{M}_{\bar{x}}}^{2} \bar{R}(\bar{x}) \mathcal{P}_{T_{\bar{x}}}$, then from (A.7) that

$$
\begin{align*}
& H_{\bar{R}}\left(x_{k}-x_{k-1}\right)+\left(H_{\bar{R}, k-1}-H_{\bar{R}}\right)\left(x_{k}-x_{k-1}\right)  \tag{A.8}\\
& =\mathcal{P}_{T_{\bar{x}}}\left(u_{k}-u_{k-1}\right)+\left(H_{\bar{R}, k-1}-H_{\bar{R}}\right)\left(u_{k}-u_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right)
\end{align*}
$$

Owing to continuity, we have $H_{\bar{R}, k-1} \rightarrow H_{\bar{R}}$ and $\mathcal{P}_{T_{x_{k-1}}} \rightarrow \mathcal{P}_{T_{\bar{x}}}$, and

$$
\begin{gathered}
\lim _{k \rightarrow+\infty} \frac{\left\|\left(H_{\bar{R}, k-1}-H_{\bar{R}}\right)\left(x_{k}-x_{k-1}\right)\right\|}{\left\|x_{k}-x_{k-1}\right\|} \leq \lim _{k \rightarrow+\infty} \frac{\left\|H_{\bar{R}, k-1}-H_{\bar{R}}\right\|\left\|x_{k}-x_{k-1}\right\|}{\left\|x_{k}-x_{k-1}\right\|}=\lim _{k \rightarrow+\infty}\left\|H_{\bar{R}, k-1}-H_{\bar{R}}\right\|=0 \\
\lim _{k \rightarrow+\infty} \frac{\|\left(\mathcal{P}_{\left.T_{x_{k-1}}-\mathcal{P}_{T_{\bar{x}}}\right)\left(w_{k}-w_{k-1}\right) \|}^{\left\|w_{k}-w_{k-1}\right\|} \leq \lim _{k \rightarrow+\infty} \frac{\| \mathcal{P}_{T_{x_{k-1}}-\mathcal{P}_{T_{\bar{x}}}\| \| w_{k}-w_{k-1} \|}^{\left\|w_{k}-w_{k-1}\right\|}=}{}=\lim _{k \rightarrow+\infty}\left\|\mathcal{P}_{T_{x_{k-1}}}-\mathcal{P}_{T_{\bar{x}}}\right\|=0\right.}{} \quad \lim _{k \rightarrow+\infty} \frac{\|\left(\mathcal{P}_{\left.T_{x_{k-1}}-\mathcal{P}_{T_{\bar{x}}}\right)\left(x_{k}-x_{k-1}\right) \|}^{\left\|x_{k}-x_{k-1}\right\|}=0\right.}{}=\text {, }
\end{gathered}
$$

Combining this with the definition of $u_{k}$, the fact that $x_{k}-x_{k-1}=\mathcal{P}_{T_{\bar{x}}}\left(x_{k}-x_{k-1}\right)+o\left(\left\|x_{k}-x_{k-1}\right\|\right)$ from Lemma A.4, and denoting $A_{T_{\bar{x}}}=A \circ \mathcal{P}_{T_{\bar{x}}}$, equation (A.8) can be written as

$$
\begin{align*}
H_{\bar{R}}\left(x_{k}-x_{k-1}\right) & =\mathcal{P}_{T_{\bar{x}}}\left(u_{k}-u_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right) \\
& =-\mathcal{P}_{T_{\bar{x}}}\left(A^{T}\left(A x_{k}-w_{k}\right)-A^{T}\left(A x_{k-1}-w_{k-1}\right)\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right) \\
& =-\mathcal{P}_{T_{\bar{x}}} A^{T} A\left(x_{k}-x_{k-1}\right)+\mathcal{P}_{T_{\bar{x}}} A^{T}\left(w_{k}-w_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right)  \tag{A.9}\\
& =-A_{T_{\bar{x}}}^{T} A_{T_{\bar{x}}}\left(x_{k}-x_{k-1}\right)+A_{T_{\bar{x}}}^{T}\left(w_{k}-w_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right)
\end{align*}
$$

Since $A$ has full rank, so is $A_{T_{\bar{x}}}$. Hence $A_{T_{\bar{x}}}^{T} A_{T_{\bar{x}}}$ is invertible and from above we have

$$
\left(\operatorname{Id}+\left(A_{T_{\bar{x}}}^{T} A_{T_{\bar{x}}}\right)^{-1} H_{\bar{R}}\right)\left(x_{k}-x_{k-1}\right)=\left(A_{T_{\bar{x}}}^{T} A_{T_{\bar{x}}}\right)^{-1} A_{T_{\bar{x}}}^{T}\left(w_{k}-w_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right)
$$

Denote $M_{\bar{R}}=A_{T_{\bar{x}}}\left(\operatorname{Id}+\left(A_{T_{\bar{x}}}^{T} A_{T_{\bar{x}}}\right)^{-1} H_{\bar{R}}\right)^{-1}\left(A_{T_{\bar{x}}}^{T} A_{T_{\bar{x}}}\right)^{-1} A_{T_{\bar{x}}}^{T}$, then

$$
\begin{equation*}
A_{T_{\bar{x}}}\left(x_{k}-x_{k-1}\right)=M_{\bar{R}}\left(w_{k}-w_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right), \tag{A.10}
\end{equation*}
$$

which concludes the proof.

## B Trajectory of ADMM

In this section, we first provide the fixed-point characterisation of ADMM based on the equivalence between ADMM and Douglas-Rachford, and then present the proofs for the trajectory of ADMM.

## B. 1 Fixed-point characterisation and convergence of ADMM

It is well-known that ADMM is equivalent to applying Douglas-Rachford splitting [17] to solve the dual problem of ( $\mathcal{P}$ ) which reads

$$
\max _{\psi \in \mathbb{R}^{p}}-\left(R^{*}\left(-A^{T} \psi\right)+J^{*}\left(-B^{T} \psi\right)+\langle\psi, b\rangle\right)
$$

( $\mathcal{D}_{\text {ADMM }}$ )
where $R^{*}(v) \stackrel{\text { def }}{=} \sup _{x \in \mathbb{R}^{n}}(\langle x, v\rangle-R(x))$ is called the Fenchel conjugate, or simply conjugate, of $R$. Below we first recall the equivalence between ADMM and Douglas-Rachford which was first established in [20], and then use the convergence of Douglas-Rachford splitting method which is well established in the literature [5] to conclude the convergence of ADMM.

- For the update of $x_{k}$, denote $u_{k}=\gamma\left(A x_{k}+B y_{k-1}-b\right)+\psi_{k-1}$ and $z_{k}=\psi_{k}-\gamma B y_{k}+\gamma b$. Since $A$ has full column rank, we have $x_{k}$ is the unique minimiser of $R(x)+\frac{\gamma}{2}\left\|A x+B y_{k-1}-b+\frac{1}{\gamma} \psi_{k-1}\right\|^{2}$. Let $R^{*}$ be the conjugate of $R$, then owing to duality, we get

$$
\begin{aligned}
& x_{k}=\operatorname{argmin}_{x \in \mathbb{R}^{n}} R(x)+\frac{\gamma}{2}\left\|A x+B y_{k-1}-b+\frac{1}{\gamma} \psi_{k-1}\right\|^{2} \\
\Longleftrightarrow & 0 \in \partial R\left(x_{k}\right)+\gamma A^{T}\left(A x_{k}+B y_{k-1}-b+\frac{1}{\gamma} \psi_{k-1}\right) \\
\Longleftrightarrow & -A^{T} u_{k} \in \partial R\left(x_{k}\right) \\
\Longleftrightarrow & x_{k} \in \partial R^{*}\left(-A^{T} u_{k}\right) \\
\Longleftrightarrow & u_{k}-\gamma A x_{k} \in u_{k}+\gamma \partial\left(R^{*} \circ-A^{T}\right)\left(u_{k}\right) \\
\Longleftrightarrow & u_{k}=\left(\operatorname{Id}+\gamma \partial\left(R^{*} \circ-A^{T}\right)\right)^{-1}\left(u_{k}-\gamma A x_{k}\right) \\
\Longleftrightarrow & u_{k}=\left(\operatorname{Id}+\gamma \partial\left(R^{*} \circ-A^{T}\right)\right)^{-1}\left(2 \psi_{k-1}-z_{k-1}\right)
\end{aligned}
$$

- For the update of $y_{k}$, the full column rank of $B$ also ensures that $y_{k}$ is the unique minimiser of $J(y)+\frac{\gamma}{2} \| A x_{k}+$ $B y-b+\frac{1}{\gamma} \psi_{k-1} \|^{2}$. Since $\psi_{k}=\psi_{k-1}+\gamma\left(A x_{k}+B y_{k}-b\right)$, then

$$
\begin{aligned}
& y_{k+1}=\operatorname{argmin}_{y \in \mathbb{R}^{m}} J(y)+\frac{\gamma}{2}\left\|A x_{k+1}+B y-b+\frac{1}{\gamma} \psi_{k-1}\right\|^{2} \\
\Longleftrightarrow & 0 \in \partial J\left(y_{k}\right)+\gamma B^{T}\left(A x_{k}+B y_{k}-b+\frac{1}{\gamma} \psi_{k-1}\right) \\
\Longleftrightarrow & -B^{T} \psi_{k} \in \partial J\left(y_{k}\right) \\
\Longleftrightarrow & y_{k} \in \partial J^{*}\left(-B^{T} \psi_{k}\right) \\
\Longleftrightarrow & \psi_{k}-\gamma B y_{k} \in \psi_{k}+\gamma \partial\left(J^{*} \circ-B^{T}\right)\left(\psi_{k}\right) \\
\Longleftrightarrow & \psi_{k}=\left(\operatorname{Id}+\gamma \partial\left(J^{*} \circ-B^{T}\right)\right)^{-1}\left(\psi_{k}-\gamma B y_{k}\right) \\
\Longleftrightarrow & \psi_{k}=\left(\operatorname{Id}+\gamma \partial\left(J^{*} \circ-B^{T}\right)\right)^{-1}\left(z_{k}-\gamma b\right) .
\end{aligned}
$$

- Summing up the above two relations we get

$$
\begin{align*}
u_{k} & =\left(\operatorname{Id}+\gamma \partial\left(R^{*} \circ-A^{T}\right)\right)^{-1}\left(2 \psi_{k-1}-z_{k-1}\right) \\
z_{k} & =z_{k-1}+u_{k}-\psi_{k-1}  \tag{B.1}\\
\psi_{k} & =\left(\operatorname{Id}+\gamma \partial\left(J^{*} \circ-B^{T}\right)\right)^{-1}\left(z_{k}-\gamma b\right)
\end{align*}
$$

which is exactly the iteration of Douglas-Rachford splitting algorithm when applied to solving the dual problem ( $\mathcal{D}_{\text {ADMM }}$ ).
Define the following operator

$$
\mathcal{F}=\frac{1}{2} \operatorname{Id}+\frac{1}{2}\left(2\left(\operatorname{Id}+\gamma \partial\left(R^{*} \circ-A^{T}\right)\right)^{-1}-\operatorname{Id}\right)\left(2\left(\operatorname{Id}+\gamma \partial\left(J^{*} \circ-B^{T}\right)\right)^{-1}-\operatorname{Id}\right)
$$

then (B.1) can be written as the fixed-point iteration in terms of $z_{k}$, that is

$$
z_{k}=\mathcal{F}\left(z_{k-1}\right)
$$

It should be noted that for $z_{k}$ we have $z_{k}=\psi_{k}-\gamma B y_{k}+\gamma b=\psi_{k-1}+\gamma A x_{k}$ which is the same as in (1.2). Owing to [5], we have that $\mathcal{F}$ is firmly non-expansive with the set of fixed-points fix $(\mathcal{F})$ being non-empty, and there exists a fixedpoint $z^{\star} \in \operatorname{fix}(\mathcal{F})$ such that $z_{k} \rightarrow z^{\star}$ which concludes the convergence of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$. Then we have $u_{k}, \psi_{k}$ converging to $\psi^{\star}=\left(\operatorname{Id}+\gamma \partial\left(J^{*} \circ-B^{T}\right)\right)^{-1}\left(z^{\star}-\gamma b\right)$ which is a dual solution of the problem $\left(\mathcal{D}_{\text {ADMM }}\right)$. The convergence of the primal ADMM sequences $\left\{x_{k}\right\}_{k \in \mathbb{N}}$ and $\left\{y_{k}\right\}_{k \in \mathbb{N}}$ follows immediately.

Owing to the above equivalence between ADMM and Douglas-Rachford splitting, we get the following relations

$$
\begin{align*}
\left\|z_{k}-z_{k-1}\right\| & \leq\left\|z_{k-1}-z_{k-2}\right\| \\
\left\|\psi_{k}-\psi_{k-1}\right\| & \leq\left\|z_{k}-z_{k-1}\right\| \leq\left\|z_{k-1}-z_{k-2}\right\|, \\
\left\|u_{k}-u_{k-1}\right\| & \leq\left\|2 \psi_{k-1}-z_{k-1}-2 \psi_{k-2}+z_{k-2}\right\| \leq 3\left\|z_{k-1}-z_{k-2}\right\|,  \tag{B.2}\\
\gamma\left\|A x_{k}-A x_{k-1}\right\| & \leq\left\|z_{k}-z_{k-1}\right\|+\left\|\psi_{k-1}-\psi_{k-2}\right\| \leq 2\left\|z_{k-1}-z_{k-2}\right\|, \\
\gamma\left\|B y_{k}-B y_{k-1}\right\| & \leq\left\|z_{k}-z_{k-1}\right\|+\left\|\psi_{k}-\psi_{k-1}\right\| \leq 2\left\|z_{k-1}-z_{k-2}\right\|,
\end{align*}
$$

which are needed in the proofs below.

## B. 2 Trajectory of ADMM: both $R, J$ are non-smooth

Given a saddle point $\left(x^{\star}, y^{\star}, \psi^{\star}\right)$ of $\mathcal{L}(x, y ; \psi)$, the first-order optimality condition entails $-A^{T} \psi^{\star} \in \partial R\left(x^{\star}\right)$ and $-B^{T} \psi^{\star} \in \partial J\left(y^{\star}\right)$. Below we impose a stronger condition

$$
\begin{equation*}
-A^{T} \psi^{\star} \in \operatorname{ri}\left(\partial R\left(x^{\star}\right)\right) \quad \text { and } \quad-B^{T} \psi^{\star} \in \operatorname{ri}\left(\partial J\left(y^{\star}\right)\right) \tag{ND}
\end{equation*}
$$

Suppose $R \in \operatorname{PSF}_{x^{\star}}\left(\mathcal{M}_{x^{\star}}^{R}\right), J \in \operatorname{PSF}_{y^{\star}}\left(\mathcal{M}_{y^{\star}}^{J}\right)$ are partly smooth, denote $T_{x^{\star}}^{R}, T_{y^{\star}}^{J}$ the tangent spaces of $\mathcal{M}_{x^{\star}}^{R}, \mathcal{M}_{y^{\star}}^{J}$ at $x^{\star}, y^{\star}$, respectively. Define the following smooth perturbation of $R, J$,

$$
\begin{equation*}
\bar{R}(x) \stackrel{\text { def }}{=} \frac{1}{\gamma}\left(R(x)-\left\langle x,-A^{T} \psi^{\star}\right\rangle\right), \bar{J}(y) \stackrel{\text { def }}{=} \frac{1}{\gamma}\left(J(y)-\left\langle w,-B^{T} \psi^{\star}\right\rangle\right) \tag{B.3}
\end{equation*}
$$

their Riemannian Hessian $H_{\bar{R}} \stackrel{\text { def }}{=} \mathcal{P}_{T_{x^{\star}}^{R}} \nabla_{\mathcal{M}_{x^{\star}}^{R}}^{2} \bar{R}\left(x^{\star}\right) \mathcal{P}_{T_{x^{\star}}^{R}}, H_{\bar{J}} \stackrel{\text { def }}{=} \mathcal{P}_{T_{x^{\star}}^{J}} \nabla_{\mathcal{M}_{y^{\star}}^{J}}^{2} \bar{J}\left(y^{\star}\right) \mathcal{P}_{T_{x^{\star}}^{J}}$ and

$$
\begin{gather*}
M_{\bar{R}} \stackrel{\text { def }}{=} A_{R}\left(\operatorname{Id}+\left(A_{R}^{T} A_{R}\right)^{-1} H_{\bar{R}}\right)^{-1}\left(A_{R}^{T} A_{R}\right)^{-1} A_{R}^{T}, \\
M_{\bar{J}} \stackrel{\text { def }}{=} B_{J}\left(\operatorname{Id}+\left(B_{J}^{T} B_{J}\right)^{-1} H_{\bar{J}}\right)^{-1}\left(B_{J}^{T} B_{J}\right)^{-1} B_{J}^{T}, \tag{B.4}
\end{gather*}
$$

where $A_{R} \stackrel{\text { def }}{=} A \circ \mathcal{P}_{T_{x^{\star}}^{R}}, B{ }_{J} \stackrel{\text { def }}{=} B \circ \mathcal{P}_{T_{y^{\star}}^{J}}$. Finally, define

$$
\begin{equation*}
M_{\mathrm{ADMM}} \stackrel{\text { def }}{=} \frac{1}{2} \mathrm{Id}+\frac{1}{2}\left(2 M_{\bar{R}}-\mathrm{Id}\right)\left(2 M_{\bar{J}}-\mathrm{Id}\right) . \tag{B.5}
\end{equation*}
$$

Proof of Theorem 2.2. The proof of Theorem 2.2 is split into several steps: finite manifold identification of ADMM, local linearisation based on partial smoothness, spectral properties of the linearised matrix, and the trajectory of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$. Let $\left(x^{\star}, y^{\star}, \psi^{\star}\right)$ be a saddle-point of $\mathcal{L}(x, y ; \psi)$.

1. Finite manifold identification of ADMM The finite manifold identification of ADMM is already discussed in [28], below we present a short discussion for the sake of self-consistency. At convergence of ADMM, owing to (1.2) we have

$$
A^{T} \psi^{\star}=\gamma A^{T}\left(A x^{\star}-\frac{1}{\gamma}\left(z^{\star}-2 \psi^{\star}\right)\right) \quad \text { and } \quad B^{T} \psi^{\star}=\gamma B^{T}\left(B y^{\star}-\frac{1}{\gamma}\left(z^{\star}-\gamma b\right)\right)
$$

From the update of $x_{k}, y_{k}$ in (1.2), we have the following monotone inclusions

$$
\begin{aligned}
-\gamma A^{T}\left(A x_{k}-\frac{1}{\gamma}\left(z_{k-1}-2 \psi_{k-1}\right)\right) & \in \partial R\left(x_{k}\right) \text { and }-\gamma B^{T}\left(B y_{k}-\frac{1}{\gamma}\left(z_{k}-\gamma b\right)\right) \in \partial J\left(y_{k}\right), \\
-\gamma A^{T}\left(A x^{\star}-\frac{1}{\gamma}\left(z^{\star}-2 \psi^{\star}\right)\right) & \in \partial R\left(x^{\star}\right) \text { and }-\gamma B^{T}\left(B y^{\star}-\frac{1}{\gamma}\left(z^{\star}-\gamma b\right)\right) \in \partial J\left(y^{\star}\right) .
\end{aligned}
$$

Since $A$ is bounded, it then follows that

$$
\begin{aligned}
\operatorname{dist}\left(-A^{T} \psi^{\star}, \partial R\left(x_{k}\right)\right) & \leq \gamma\left\|A^{T}\left(A x_{k}-\frac{1}{\gamma}\left(z_{k-1}-2 \psi_{k-1}\right)\right)-A^{T}\left(A x^{\star}-\frac{1}{\gamma}\left(z^{\star}-2 \psi^{\star}\right)\right)\right\| \\
& \leq \gamma\|A\|\left\|A\left(x_{k}-x^{\star}\right)-\frac{1}{\gamma}\left(z_{k-1}-z^{\star}\right)+\frac{2}{\gamma}\left(\psi_{k-1}-\psi^{\star}\right)\right\| \\
& \leq \gamma\|A\|\left(\|A\|\left\|x_{k}-x^{\star}\right\|+\frac{1}{\gamma}\left\|z_{k-1}-z^{\star}\right\|+\frac{2}{\gamma}\left\|\psi_{k-1}-\psi^{\star}\right\|\right) \rightarrow 0 .
\end{aligned}
$$

and similarly

$$
\operatorname{dist}\left(-B^{T} \psi^{\star}, \partial J\left(y_{k}\right)\right) \leq \gamma\|B\|\left(\|B\|\left\|y_{k}-y^{\star}\right\|+\frac{1}{\gamma}\left\|z_{k}-z^{\star}\right\|\right) \rightarrow 0
$$

Since $R \in \Gamma_{0}\left(\mathbb{R}^{n}\right)$ and $J \in \Gamma_{0}\left(\mathbb{R}^{m}\right)$, then by the sub-differentially continuous property of them we have $R\left(x_{k}\right) \rightarrow R\left(x^{\star}\right)$ and $J\left(y_{k}\right) \rightarrow J\left(y^{\star}\right)$. Hence the conditions of [21, Theorem 5.3] are fulfilled for $R$ and $J$, and there exists $K$ large enough such that for all $k \geq K$, there holds

$$
\left(x_{k}, y_{k}\right) \in \mathcal{M}_{x^{\star}}^{R} \times \mathcal{M}_{y^{\star}}^{J}
$$

which is the finite manifold identification.
2. Linearisation of ADMM For convenience, denote $\beta=1 / \gamma$. For the update of $y_{k}$, define $w_{k}=-\beta\left(z_{k}-\gamma b\right)$, we have from (1.2) that

$$
y_{k}=\operatorname{argmin}_{y \in \mathbb{R}^{m}} \beta J(y)+\frac{1}{2}\left\|B y-w_{k}\right\|^{2}
$$

Owing to the optimality condition of a saddle point, define $\bar{J}(y) \stackrel{\text { def }}{=} \beta J(y)-\left\langle y,-\beta B^{T} \psi^{\star}\right\rangle$ and its Riemannian Hessian $H_{\bar{J}}=\mathcal{P}_{T_{y^{\star}}^{J}} \nabla_{\mathcal{M}_{y^{\star}}^{J}}^{2} \bar{J}\left(y^{\star}\right) \mathcal{P}_{T_{y^{\star}}^{J}}$. For $B$, define $B_{J}=B \circ \mathcal{P}_{T_{y^{\star}}^{J}}$, and

$$
M_{\bar{J}}=B_{J}\left(\operatorname{Id}+\left(B_{J}^{T} B_{J}\right)^{-1} H_{\bar{J}}\right)^{-1}\left(B_{J}^{T} B_{J}\right)^{-1} B_{J}^{T}
$$

Then owing to Lemma A.9, we get

$$
\begin{align*}
B_{J}\left(y_{k}-y_{k-1}\right) & =M_{\bar{J}}\left(w_{k}-w_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right)  \tag{B.6}\\
& =-\beta M_{\bar{J}}\left(z_{k}-z_{k-1}\right)+o\left(\left\|z_{k}-z_{k-1}\right\|\right) .
\end{align*}
$$

Now we turn to $x_{k}$. Define $w_{k}=\beta\left(z_{k-1}-2 \psi_{k-1}\right)$, then we get from (1.2) that

$$
x_{k}=\operatorname{argmin}_{x \in \mathbb{R}^{n}} \beta R(x)+\frac{1}{2}\left\|A x-w_{k}\right\|^{2} .
$$

Define $\bar{R}(x) \stackrel{\text { def }}{=} \beta R(x)-\left\langle x,-\beta A^{T} \psi^{\star}\right\rangle$ and $H_{\bar{R}}=\mathcal{P}_{T_{x^{\star}}^{R}} \nabla_{\mathcal{M}_{x^{\star}}^{R}}^{2} \bar{R}\left(x^{\star}\right) \mathcal{P}_{T_{x^{\star}}^{R}}$. Denote $A_{R}=A \circ \mathcal{P}_{T_{x^{\star}}^{R}}$, and

$$
M_{\bar{R}}=A_{R}\left(\operatorname{Id}+\left(A_{R}^{T} A_{R}\right)^{-1} H_{\bar{R}}\right)^{-1}\left(A_{R}^{T} A_{R}\right)^{-1} A_{R}^{T}
$$

Note from (1.2) that $\psi_{k-1}-\psi_{k-2}=z_{k-1}-z_{k-2}+\gamma B\left(y_{k-1}-y_{k-2}\right)$, then

$$
\begin{aligned}
w_{k}-w_{k-1} & =\beta\left(z_{k-1}-z_{k-2}\right)-2 \beta\left(\psi_{k-1}-\psi_{k-2}\right) \\
& =-\beta\left(z_{k-1}-z_{k-2}\right)-2 \beta \gamma B\left(y_{k-1}-y_{k-2}\right) \\
& =-\beta\left(z_{k-1}-z_{k-2}\right)-2 B_{J}\left(y_{k-1}-y_{k-2}\right)+o\left(\left\|y_{k-1}-y_{k-2}\right\|\right)
\end{aligned}
$$

where $y_{k-1}-y_{k-2}=\mathcal{P}_{T_{x^{\star}}^{R}}\left(y_{k-1}-y_{k-2}\right)+o\left(\left\|y_{k-1}-y_{k-2}\right\|\right)$ from Lemma A. 4 is applied. From (B.2), we have $o\left(\left\|y_{k-1}-y_{k-2}\right\|\right)=o\left(\left\|z_{k-1}-z_{k-2}\right\|\right)$ and $o\left(\left\|w_{k-1}-w_{k-2}\right\|\right)=o\left(\left\|z_{k-1}-z_{k-2}\right\|\right)$, then applying Lemma A. 9 yields,

$$
\begin{align*}
A_{R}\left(x_{k}-x_{k-1}\right) & =M_{\bar{R}}\left(w_{k}-w_{k-1}\right)+o\left(\left\|w_{k}-w_{k-1}\right\|\right) \\
& =-\beta M_{\bar{R}}\left(z_{k-1}-z_{k-2}\right)+2 M_{\bar{R}} B_{J}\left(y_{k-1}-y_{k-2}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right)  \tag{B.7}\\
& =-\beta M_{\bar{R}}\left(z_{k-1}-z_{k-2}\right)+2 \beta M_{\bar{R}} M_{\bar{J}}\left(z_{k-1}-z_{k-2}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right)
\end{align*}
$$

Finally, from (1.2), (B.6) and (B.7), we have that

$$
\begin{aligned}
z_{k}-z_{k-1}= & \left(z_{k-1}+\gamma\left(A x_{k}+B y_{k-1}-b\right)\right)-\left(z_{k-2}+\gamma\left(A x_{k-1}+B y_{k-2}-b\right)\right) \\
= & \left(z_{k-1}-z_{k-2}\right)+\gamma A\left(x_{k}-x_{k-1}\right)+\gamma B\left(y_{k-1}-y_{k-2}\right) \\
= & \left(z_{k-1}-z_{k-2}\right)+\gamma A_{R}\left(x_{k}-x_{k-1}\right)+\gamma B_{J}\left(y_{k-1}-y_{k-2}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right) \\
= & \left(z_{k-1}-z_{k-2}\right)-M_{\bar{R}}\left(z_{k-1}-z_{k-2}\right)+2 M_{\bar{R}} M_{\bar{J}}\left(z_{k-1}-z_{k-2}\right)+M_{\bar{J}}\left(z_{k-1}-z_{k-2}\right) \\
& \quad+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right) \\
= & \left(\operatorname{Id}+2 M_{\bar{R}} M_{\bar{J}}-M_{\bar{R}}-M_{\bar{J}}\right)\left(z_{k-1}-z_{k-2}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right),
\end{aligned}
$$

which is the desired linearisation of ADMM.
3. Spectral properties of $M_{\mathrm{ADMM}}$ Consider first the case where both $R, J$ are general partly smooth functions, under which we can shown the non-expansiveness of $M_{\mathrm{ADMM}}$. For $M_{\bar{R}}$, since $A$ is injective, so is $A_{R}$, then $A_{R}^{T} A_{R}$ is symmetric positive definite. Therefore, we have the following similarity result for $M_{\bar{R}}$,

$$
\begin{align*}
M_{\bar{R}} & =A_{R}\left(\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}}\left(\operatorname{Id}+\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}} H_{\bar{R}}\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}}\right)\left(A_{R}^{T} A_{R}\right)^{\frac{1}{2}}\right)^{-1}\left(A_{R}^{T} A_{R}\right)^{-1} A_{R}^{T} \\
& =A_{R}\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}}\left(\operatorname{Id}+\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}} H_{\bar{R}}\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}}\right)^{-1}\left(A_{R}^{T} A_{R}\right)^{\frac{1}{2}}\left(A_{R}^{T} A_{R}\right)^{-1} A_{R}^{T}  \tag{B.8}\\
& =A_{R}\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}}\left(\operatorname{Id}+\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}} H_{\bar{R}}\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}}\right)^{-1}\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}} A_{R}^{T} .
\end{align*}
$$

Since $\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}} H_{\bar{R}}\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}}$ is symmetric positive definite, hence maximal monotone, then the matrix

$$
\left(\operatorname{Id}+\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}} H_{\bar{R}}\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}}\right)^{-1}
$$

is firmly non-expansive. Let $A_{R}=U S V^{T}$ be the SVD of $A_{R}$, then we have

$$
\left\|A_{R}\left(A_{R}^{T} A_{R}\right)^{-\frac{1}{2}}\right\|=\left\|U S V^{T}\left(V S U^{T} U S V^{T}\right)^{-\frac{1}{2}}\right\|=\left\|U S V^{T}\left(V S^{2} V^{T}\right)^{-\frac{1}{2}}\right\|=\left\|U S V^{T} V S^{-1} V^{T}\right\|=1
$$

Then owing to [5, Example 4.14], $M_{\bar{R}}$ is firmly non-expansive. Similarly, $M_{\bar{J}}$ is firmly non-expansive, and so is $M_{\text {admm }}$ [5, Proposition 4.31]. Therefore, the power $M_{\mathrm{ADMM}}^{k}$ is convergent.

Now suppose that both $R, J$ are locally polyhedral around $\left(x^{\star}, y^{\star}\right)$, then $M_{\bar{R}}$ and $M_{\bar{J}}$ become

$$
M_{\bar{R}}=A_{R}\left(A_{R}^{T} A_{R}\right)^{-1} A_{R}^{T} \quad \text { and } \quad M_{\bar{J}}=B_{J}\left(B_{J}^{T} B_{J}\right)^{-1} B_{J}^{T}
$$

which are projection operators onto the ranges of $A_{R}$ and $B_{J}$ respectively. Denote these two subspaces by $T_{A_{R}}$ and $T_{B_{J}}$, and correspondingly $\mathcal{P}_{T_{A_{R}}} \stackrel{\text { def }}{=} A_{R}\left(A_{R}^{T} A_{R}\right)^{-1} A_{R}^{T}$ and $\mathcal{P}_{T_{B}} \stackrel{\text { def }}{=} B_{J}\left(B_{J}^{T} B_{J}\right)^{-1} B_{J}^{T}$. Then

$$
M_{\mathrm{ADMM}}=\mathcal{P}_{T_{A_{R}}} \mathcal{P}_{T_{B_{J}}}+\left(\mathrm{Id}-\mathcal{P}_{T_{A_{R}}}\right)\left(\mathrm{Id}-\mathcal{P}_{T_{B_{J}}}\right) .
$$

Denote the dimension of $T_{A_{R}}, T_{B_{J}}$ by $\operatorname{dim}\left(T_{A_{R}}\right)=p, \operatorname{dim}\left(T_{B_{J}}\right)=q$, and the dimension of the intersection $\operatorname{dim}\left(T_{A_{R}} \cap\right.$ $\left.T_{B_{J}}\right)=d$. Without the loss of generality, we assume that $1 \leq p \leq q \leq n$. Consequently, there are $r=p-d$ principal angles $\left(\zeta_{i}\right)_{i=1, \ldots, r}$ between $T_{A_{R}}$ and $T_{B_{J}}$ that are strictly greater than 0 and smaller than $\pi / 2$. Suppose that $\zeta_{1} \leq \cdots \leq \zeta_{r}$. Define the following two diagonal matrices

$$
C=\operatorname{diag}\left(\cos \left(\zeta_{1}\right), \cdots, \cos \left(\zeta_{r}\right)\right) \quad \text { and } \quad S=\operatorname{diag}\left(\sin \left(\zeta_{1}\right), \cdots, \sin \left(\zeta_{r}\right)\right)
$$

Owing to [7, 16], there exists a real orthogonal matrix $U$ such that

$$
M_{\mathrm{ADMM}}=U\left[\begin{array}{cc|cc}
C^{2} & C S & 0 & 0 \\
-C S & C^{2} & 0 & 0 \\
\hline 0 & 0 & 0_{q-p+2 d} & 0 \\
0 & 0 & 0 & \mathrm{Id}_{n-p-q}
\end{array}\right] U^{T}
$$

which indicates $M_{\text {ADMM }}$ is normal and all its eigenvalues are inside unit disc.
Let $M_{\mathrm{ADMM}}^{\infty}=\lim _{k \rightarrow+\infty} M_{\mathrm{ADMM}}^{k}$ and $\widetilde{M}_{\mathrm{ADMM}}=M_{\mathrm{ADMM}}-M_{\mathrm{ADMM}}^{\infty}$, then we have

$$
\widetilde{M}_{\mathrm{ADMM}}=U\left[\begin{array}{cc|c}
C^{2} & C S & 0  \tag{B.9}\\
-C S & C^{2} & 0 \\
\hline 0 & 0 & 0_{n-2 r}
\end{array}\right] U^{T}
$$

4. Trajectory of ADMM Owing to the polyhedrality of $R$ and $J$, all the small $o$-terms in the linearisation proof vanish and we get directly

$$
\begin{equation*}
z_{k}-z_{k-1}=M_{\mathrm{ADMM}}\left(z_{k-1}-z_{k-2}\right)=M_{\mathrm{ADMM}}^{k}\left(z_{0}-z_{-1}\right) \tag{B.10}
\end{equation*}
$$

As $v_{k} \stackrel{\text { def }}{=} z_{k}-z_{k-1} \rightarrow 0$, passing to the limit we get from above

$$
0=\lim _{k \rightarrow+\infty} M_{\mathrm{ADMM}}^{k} v_{0}=M_{\mathrm{ADMM}}^{\infty} v_{0}
$$

which means $v_{0} \in \operatorname{ker}\left(M_{\mathrm{ADMM}}\right)$ where $\operatorname{ker}\left(M_{\mathrm{ADMM}}\right)$ denotes the kernel of $M_{\mathrm{ADMM}}$. Since $M_{\mathrm{ADMM}}^{\infty} M_{\mathrm{ADMM}}^{k}=M_{\mathrm{ADMM}}^{\infty}$, we have $v_{k} \in \operatorname{ker}\left(M_{\mathrm{ADMM}}\right)$ holds for any $k \in \mathbb{N}$. Then from (B.10) we have

$$
v_{k}=\left(M_{\mathrm{ADMM}}-M_{\mathrm{ADMM}}^{\infty}\right) v_{k}=\widetilde{M}_{\mathrm{ADMM}} v_{k-1} .
$$

The block diagonal property of (B.9) indicates that there exists an elementary transformation matrix $E$ such that

$$
\widetilde{M}_{\mathrm{ADMM}}=U E\left[\begin{array}{llll}
B_{1} & & & \\
& \ddots & & \\
& & B_{r} & \\
& & & 0_{n-2 r}
\end{array}\right] E U^{T}
$$

where for each $i=1, \ldots, r$, we have

$$
B_{i}=\cos \left(\zeta_{i}\right)\left[\begin{array}{cc}
\cos \left(\zeta_{i}\right) & \sin \left(\zeta_{i}\right) \\
-\sin \left(\zeta_{i}\right) & \cos \left(\zeta_{i}\right)
\end{array}\right]
$$

which is rotation matrix scaled by $\cos \left(\zeta_{i}\right)$. It is easy to show that, for each $i=1, \ldots, d$, there holds

$$
\lim _{k \rightarrow+\infty} B_{i}^{k}=0
$$

since the spectral radius of $B_{i}$ is $\rho\left(B_{i}\right)=\cos \left(\zeta_{i}\right)<1$.
Suppose for some $1 \leq e<r$, we have

$$
\zeta=\zeta_{1}=\cdots=\zeta_{e}<\zeta_{e+1} \leq \cdots \leq \zeta_{r}
$$

Consider the following decompositions

$$
\Gamma_{1}=\left[\begin{array}{cccc}
B_{1} & & & \\
& \ddots & & \\
& & B_{e} & \\
& & & 0_{n-2 e}
\end{array}\right] \text { and } \Gamma_{2}=\left[\begin{array}{cccc}
B_{1} & & & \\
& \ddots & & \\
& & B_{r} & \\
& & & 0_{n-2 r}
\end{array}\right]-\Gamma_{1}
$$

Denote $\eta=\frac{\cos \left(\zeta_{e+1}\right)}{\cos (\zeta)}$, it is immediate to see that $\frac{1}{\cos ^{k}(\zeta)} \Gamma_{2}^{k}=O\left(\eta^{k}\right) \rightarrow 0$, and for each $i=1, \ldots, e$

$$
\frac{1}{\cos (\zeta)} B_{i}=\left[\begin{array}{cc}
\cos (\zeta) & \sin (\zeta) \\
-\sin (\zeta) & \cos (\zeta)
\end{array}\right]
$$

which is a circular rotation. Therefore, $\frac{1}{\cos (\zeta)} \Gamma_{1}$ is a rotation with respect to the first $2 e$ elements. Denote $u_{k}=E U^{T} v_{k}$, then from $v_{k}=\widetilde{M} v_{k-1}=U E\left(\Gamma_{1}+\Gamma_{2}\right) E U^{T} v_{k}$, we get

$$
u_{k}=\left(\Gamma_{1}+\Gamma_{2}\right) u_{k}=\left(\Gamma_{1}+\Gamma_{2}\right)^{k} u_{0}=\Gamma_{1}^{k} u_{0}+\Gamma_{2}^{k} u_{0}
$$

which is an orthogonal decomposition of $u_{k}$. Define

$$
s_{k}=\frac{1}{\cos ^{k}(\zeta)} \Gamma_{1}^{k} u_{1} \quad \text { and } \quad t_{k}=\frac{1}{\cos ^{k}(\zeta)} \Gamma_{2}^{k} u_{1}
$$

then we have that $\left\|s_{k}\right\|=\left\|s_{k-1}\right\|$ and $\left\langle s_{k}, s_{k-1}\right\rangle=\cos (\zeta)\left\|s_{k}\right\|^{2}$, and $t_{k}=O\left(\eta^{k}\right)$. As a result, for $\cos \left(\theta_{k}\right)$ we have

$$
\begin{align*}
\cos \left(\theta_{k}\right)=\frac{\left\langle v_{k}, v_{k-1}\right\rangle}{\left\|v_{k}\right\|\left\|v_{k-1}\right\|}=\frac{\left\langle u_{k}, u_{k-1}\right\rangle}{\left\|u_{k}\right\|\left\|u_{k-1}\right\|} & =\frac{\left\langle s_{k}+t_{k}, s_{k-1}+t_{k-1}\right\rangle}{\left\|s_{k}+t_{k}\right\|\left\|s_{k-1}+t_{k-1}\right\|} \\
& =\frac{\left\langle s_{k}, s_{k-1}\right\rangle}{\left\|s_{k}+t_{k}\right\|\left\|s_{k-1}+t_{k-1}\right\|}+\frac{\left\langle t_{k}, t_{k-1}\right\rangle}{\left\|s_{k}+t_{k}\right\|\left\|s_{k-1}+t_{k-1}\right\|}  \tag{B.11}\\
& =\frac{\left\|s_{k}\right\|^{2} \cos (\zeta)}{\left\|s_{k}\right\|^{2}+\left\|t_{k}\right\|^{2}} \cdot \frac{\left\|s_{k}+t_{k}\right\|}{\left\|s_{k-1}+t_{k-1}\right\|}+O\left(\eta^{2 k-1}\right)
\end{align*}
$$

Using the fact that

$$
\frac{\left\|s_{k}\right\|^{2} \cos (\zeta)}{\left\|s_{k}\right\|^{2}+\left\|t_{k}\right\|^{2}}=\cos (\zeta)\left(1-\left\|t_{k}\right\|^{2}+O\left(\left\|t_{k}\right\|^{4}\right)\right)=\cos (\zeta)+O\left(\eta^{2 k}\right) \quad \text { and } \quad \frac{\left\|s_{k}+t_{k}\right\|}{\left\|s_{k-1}+t_{k-1}\right\|} \rightarrow 1
$$

we conclude that $\cos \left(\theta_{k}\right) \rightarrow \cos (\zeta)$. As a matter of fact, we have $\cos \left(\theta_{k}\right)-\cos (\zeta)=O\left(\eta^{2 k}\right)$ which shows how fast $\cos \left(\theta_{k}\right)$ converges to $\cos (\zeta)$.

## B. 3 Trajectory of ADMM: $R$ or/and $J$ is smooth

Now we consider the case that at least one function out of $R, J$ is smooth. For simplicity, consider that $R$ is smooth and $J$ remains non-smooth. Assume that $R$ is locally $C^{2}$-smooth around $x^{\star}$, the Hessian of $R$ at $x^{\star}$ reads $\nabla^{2} R\left(x^{\star}\right)$ which is positive semi-definite owing to convexity. Define $M_{R} \stackrel{\text { def }}{=} A\left(\operatorname{Id}+\frac{1}{\gamma}\left(A^{T} A\right)^{-1} \nabla^{2} R\left(x^{\star}\right)\right)^{-1}\left(A^{T} A\right)^{-1} A^{T}$, and redefine

$$
\begin{equation*}
M_{\mathrm{ADMM}} \stackrel{\text { def }}{=} \frac{1}{2} \mathrm{Id}+\frac{1}{2}\left(2 M_{R}-\mathrm{Id}\right)\left(2 M_{\bar{J}}-\mathrm{Id}\right) . \tag{B.12}
\end{equation*}
$$

Proof of Proposition 2.4. We prove the corollary in two steps.

1. Linearisation of ADMM Following the above proof, we have for $y_{k}$ that

$$
B_{J}\left(y_{k}-y_{k-1}\right)=\beta M_{\bar{J}}\left(z_{k}-z_{k-1}\right)+o\left(\left\|z_{k}-z_{k-1}\right\|\right) .
$$

From (1.2), for $x_{k+1}$ and $x_{k}$, since $R$ is globally smooth differentiable

$$
-A^{T}\left(A x_{k}-\beta\left(z_{k-1}-2 \psi_{k-1}\right)\right) \in \beta \nabla R\left(x_{k}\right) \text { and }-A^{T}\left(A x_{k-1}-\beta\left(z_{k-2}-2 \psi_{k-2}\right)\right) \in \beta \nabla R\left(x_{k-1}\right)
$$

which leads to, applying the local $C^{2}$-smoothness of $R$ around $x^{\star}$

$$
\begin{aligned}
& -A^{T}\left(A x_{k}-\beta\left(z_{k-1}-2 \psi_{k-1}\right)\right)+A^{T}\left(A x_{k-1}-\beta\left(z_{k-2}-2 \psi_{k-2}\right)\right) \\
& =\beta \nabla R\left(x_{k}\right)-\beta \nabla R\left(x_{k-1}\right) \\
& =\beta \nabla^{2} R\left(x_{k-1}\right)\left(x_{k}-x_{k-1}\right)+o\left(\left\|x_{k}-x_{k-1}\right\|\right) \\
& =\beta \nabla^{2} R\left(x^{\star}\right)\left(x_{k}-x_{k-1}\right)+\beta\left(\nabla^{2} R\left(x_{k-1}\right)-\nabla^{2} R\left(x^{\star}\right)\right)\left(x_{k}-x_{k-1}\right)+o\left(\left\|x_{k}-x_{k-1}\right\|\right) \\
& =\beta \nabla^{2} R\left(x^{\star}\right)\left(x_{k}-x_{k-1}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right)
\end{aligned}
$$

Using the fact that $A^{T} A$ is invertible and rearranging terms, we arrive at

$$
\begin{aligned}
& \left(\operatorname{Id}+\beta\left(A^{T} A\right)^{-1} \nabla^{2} R\left(x^{\star}\right)\right)\left(x_{k}-x_{k-1}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right) \\
& =\beta\left(A^{T} A\right)^{-1} A^{T}\left(z_{k-1}-z_{k-2}\right)-2 \beta\left(A^{T} A\right)^{-1} A^{T}\left(\psi_{k-1}-\psi_{k-2}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right) \\
& =-\beta\left(A^{T} A\right)^{-1} A^{T}\left(z_{k-1}-z_{k-2}\right)+2\left(A^{T} A\right)^{-1} A^{T} B_{J}\left(y_{k-1}-y_{k-2}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right)
\end{aligned}
$$

which further leads to, denote $M_{R}=A\left(\operatorname{Id}+\left(A^{T} A\right)^{-1} H_{R}\right)^{-1}\left(A^{T} A\right)^{-1} A^{T}$

$$
\begin{aligned}
A\left(x_{k}-x_{k-1}\right) & =-\beta M_{R}\left(z_{k-1}-z_{k-2}\right)+2 M_{R} B_{J}\left(y_{k-1}-y_{k-2}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right) \\
& =-\beta M_{R}\left(z_{k-1}-z_{k-2}\right)+2 \beta M_{R} M_{\bar{J}}\left(z_{k-1}-z_{k-2}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right)
\end{aligned}
$$

Finally, from (1.2), we have that

$$
z_{k}-z_{k-1}=\left(\operatorname{Id}+2 M_{R} M_{\bar{J}}-M_{R}-M_{\bar{J}}\right)\left(z_{k-1}-z_{k-2}\right)+o\left(\left\|z_{k-1}-z_{k-2}\right\|\right)
$$

2. Trajectory of ADMM Since $A$ is full rank square matrix and hence invertible, from (B.8) we have

$$
\begin{aligned}
M_{R} & =A\left(\operatorname{Id}+\frac{1}{\gamma}\left(A^{T} A\right)^{-1} \nabla^{2} R\left(x^{\star}\right)\right)^{-1}\left(A^{T} A\right)^{-1} A^{T} \\
& =A\left(A^{T} A\right)^{-\frac{1}{2}}\left(\operatorname{Id}+\frac{1}{\gamma}\left(A^{T} A\right)^{-\frac{1}{2}} \nabla^{2} R\left(x^{\star}\right)\left(A^{T} A\right)^{-\frac{1}{2}}\right)^{-1}\left(A^{T} A\right)^{-\frac{1}{2}} A^{T} \\
& \sim\left(\operatorname{Id}+\frac{1}{\gamma}\left(A^{T} A\right)^{-\frac{1}{2}} \nabla^{2} R\left(x^{\star}\right)\left(A^{T} A\right)^{-\frac{1}{2}}\right)^{-1}
\end{aligned}
$$

where $\left(\operatorname{Id}+\frac{1}{\gamma}\left(A^{T} A\right)^{-\frac{1}{2}} \nabla^{2} R\left(x^{\star}\right)\left(A^{T} A\right)^{-\frac{1}{2}}\right)^{-1}$ is symmetric positive definite. If we choose $\gamma$ such that

$$
\frac{1}{\gamma}\left\|\left(A^{T} A\right)^{-\frac{1}{2}} \nabla^{2} R\left(x^{\star}\right)\left(A^{T} A\right)^{-\frac{1}{2}}\right\|<1
$$

then all the eigenvalues of $M_{R}$ are in $\left.] 1 / 2,1\right]$, hence $W_{R} \stackrel{\text { def }}{=} 2 M_{R}$-Id is symmetric positive definite. Therefore, we get

$$
\begin{aligned}
\frac{1}{2} \mathrm{Id}+\frac{1}{2} W_{R}\left(2 M_{\bar{J}}-\mathrm{Id}\right) & =W_{R}^{1 / 2}\left(\frac{1}{2} \mathrm{Id}+\frac{1}{2} W_{R}^{1 / 2}\left(2 M_{\bar{J}}-\mathrm{Id}\right) W_{R}^{1 / 2}\right) W_{R}^{-1 / 2} \\
& \sim \frac{1}{2} \operatorname{Id}+\frac{1}{2} W_{R}^{1 / 2}\left(2 M_{\bar{J}}-\mathrm{Id}\right) W_{R}^{1 / 2}
\end{aligned}
$$

and $\bar{M} \stackrel{\text { def }}{=} \frac{1}{2} \operatorname{Id}+\frac{1}{2} W_{R}^{1 / 2}\left(2 M_{\bar{J}}-\mathrm{Id}\right) W_{R}^{1 / 2}$ is symmetric positive semi-definite with all eigenvalues in $[0,1]$. Hence, by similarity, the eigenvalues of $M$ are all real and contained in $[0,1]$.

## C Adaptive acceleration for ADMM

## C. 1 Convergence of $\mathrm{A}^{3} \mathrm{DMM}$

Proof of Proposition 4.2. From (4.3), we have that

$$
z_{k}=\mathcal{F}\left(z_{k-1}+\varepsilon_{k-1}\right)=\mathcal{F}\left(z_{k-1}\right)+\left(\mathcal{F}\left(z_{k-1}+\varepsilon_{k-1}\right)-\mathcal{F}\left(z_{k-1}\right)\right)
$$

Given any $z^{\star} \in \operatorname{fix}(\mathcal{F})$, since $\mathcal{F}$ is firmly non-expansive, hence non-expansive, we have

$$
\left\|z_{k}-z^{\star}\right\| \leq\left\|\mathcal{F}\left(z_{k-1}\right)-\mathcal{F}\left(z^{\star}\right)\right\|+\left\|\mathcal{F}\left(z_{k-1}+\varepsilon_{k}\right)-\mathcal{F}\left(z_{k-1}\right)\right\| \leq\left\|z_{k-1}-z^{\star}\right\|+\left\|\varepsilon_{k-1}\right\|
$$

which means that $\left\{z_{k}\right\}_{k \in \mathbb{N}}$ is quasi-Fejér monotone with respect to $\operatorname{fix}(\mathcal{F})$. Then invoke [5, Proposition 5.34] we obtain the convergence of the sequence $\left\{z_{k}\right\}_{k \in \mathbb{N}}$.

## C. 2 Acceleration guarantee of $\mathbf{A}^{3} \mathbf{D M M}$

Recall the definition of $V_{k-1}, c_{k}, C_{k}$ and $\bar{z}_{k, s}$ in the beginning of the section. By definition,

$$
\begin{equation*}
V_{k}=M V_{k-1} \tag{C.1}
\end{equation*}
$$

Define $E_{k, j} \stackrel{\text { def }}{=} V_{k} C_{k}^{j}-V_{k+1}$ for $j \geq 1$ and

$$
E_{k, 0} \stackrel{\text { def }}{=} V_{k-1} C_{k}-V_{k}=\left[\begin{array}{llll}
\left(V_{k-1} c_{k}-v_{k}\right) & 0 & \cdots & 0 \tag{C.2}
\end{array}\right] .
$$

We obtain the relation between the extrapolated point $\bar{z}_{k, s}$ and the $(k+s)$ 'th point of $\left\{z_{k}\right\}_{k \in \mathbb{N}}$

$$
\bar{z}_{k, s}=z_{k}+\sum_{j=1}^{s}\left(v_{j+k}+\left(E_{k, j}\right)_{(:, 1)}\right)=z_{k+s}+\sum_{j=1}^{s}\left(E_{k, j}\right)_{(:, 1)}
$$

In the following, given a matrix $M$, we let $\rho(M)$ denote the spectral radius of $M$ and $\lambda(M)$ denote its spectrum.
Proof of Proposition 4.3. We first prove $(i)$ that the extrapolation error is controlled by the coefficients fitting error. Since $k \in \mathbb{N}$ is fixed, for ease of notation, we also write $E_{\ell}=E_{k, \ell}$ and $C=C_{k}$. We first show that for $\ell \in \mathbb{N}$, we have

$$
\begin{equation*}
E_{\ell}=\sum_{j=1}^{\ell} M^{j} E_{0} C^{\ell-j} \tag{C.3}
\end{equation*}
$$

We prove this by induction. Note that

$$
V_{k} C \stackrel{(\mathrm{C} .1)}{=}\left(M V_{k-1}\right) C \stackrel{(\mathrm{C} .2)}{=} M V_{k}+M E_{0} \stackrel{(\mathrm{C} .1)}{=} V_{k+1}+M E_{0} .
$$

Therefore, $E_{1}=M E_{0}$ as required. Assume that (C.4) is true up to $\ell=m$. Then,

$$
\begin{gathered}
V_{k} C^{m+1} \stackrel{(\mathrm{C} .1)}{=}\left(M V_{k-1}\right) C^{m+1} \stackrel{(\mathrm{C.2)}}{=} M V_{k} C^{m}+M E_{0} C^{m}=M\left(V_{m+k}+E_{m}\right)+M E_{0} C^{m} \\
\stackrel{(\mathrm{C} .1)}{=} V_{m+2}+M E_{m}+M E_{0} C^{m}
\end{gathered}
$$

So, plugging in our assumption on $E_{m}$, we have

$$
E_{m+1}=M E_{m}+M E_{0} C^{m}=M E_{0} C^{m}+M\left(\sum_{j=1}^{m} M^{j} E_{0} C^{m-j}\right)=\sum_{j=1}^{m+1} M^{j} E_{0} C^{m+1-j}
$$

To bound the extrapolation error,

$$
\sum_{m=1}^{s} E_{m}=\sum_{m=1}^{s}\left(\sum_{j=1}^{m} M^{j} E_{0} C^{m-j}\right)=\sum_{\ell=0}^{s-1}\left(\sum_{j=1}^{s-\ell} M^{j}\right) E_{0} C^{\ell}=\sum_{\ell=1}^{s} M^{\ell} E_{0}\left(\sum_{i=0}^{s-\ell} C^{i}\right)
$$

Therefore,

$$
\left\|\bar{z}_{k, s}-z^{\star}\right\| \leq\left\|z_{k+s}-z^{\star}\right\|+\sum_{\ell=1}^{s}\left\|M^{\ell}\right\|\left\|E_{0}\right\|\left\|\sum_{i=0}^{s-\ell} C_{(1,1)}^{i}\right\| .
$$

In the case of $s=+\infty$, we have

$$
\left\|\bar{z}_{k, \infty}-z^{\star}\right\| \leq \sum_{\ell=1}^{\infty}\left\|M^{\ell}\right\|\left\|E_{0}(\operatorname{Id}-C)_{(:, 1)}^{-1}\right\|=\frac{\left\|E_{0}\right\|}{1-\sum_{i} c_{i}} \sum_{\ell=1}^{\infty}\left\|M^{\ell}\right\|
$$

The fact that $B_{s}$ is uniformly bounded in $s$ if $\rho(M)<1$ and $\rho(C)<1$ follows because this implies that $\sum_{\ell=1}^{\infty}\left\|M^{\ell}\right\|<\infty$ thanks to the Gelfand formula, and $\sum_{i=0}^{\infty} C^{i}=(\operatorname{Id}-C)^{-1}$ and its $(1,1)^{t h}$ entry is precisely $\frac{1}{1-\sum_{i} c_{i}}$. Since $k \in \mathbb{N}$ is fixed, for ease of notation, we also write $E_{\ell}=E_{k, \ell}$ and $C=C_{k}$. We first show that for $\ell \in \mathbb{N}$, we have

$$
\begin{equation*}
E_{\ell}=\sum_{j=1}^{\ell} M^{j} E_{0} C^{\ell-j} \tag{C.4}
\end{equation*}
$$

We prove this by induction. Note that

$$
V_{k} C \stackrel{(\mathrm{C} .1)}{=}\left(M V_{k-1}\right) C \stackrel{(\mathrm{C} .2)}{=} M V_{k}+M E_{0} \stackrel{(\mathrm{C} .1)}{=} V_{k+1}+M E_{0} .
$$

Therefore, $E_{1}=M E_{0}$ as required. Assume that (C.4) is true up to $\ell=m$. Then,

$$
\begin{aligned}
V_{k} C^{m+1} & \stackrel{(\mathrm{C} .1)}{=}\left(M V_{k-1}\right) C^{m+1} \\
& \stackrel{(\mathrm{C} .2)}{=} M V_{k} C^{m}+M E_{0} C^{m}=M\left(V_{m+k}+E_{m}\right)+M E_{0} C^{m} \\
& \stackrel{(\mathrm{C} .1)}{=} V_{m+2}+M E_{m}+M E_{0} C^{m}
\end{aligned}
$$

So, plugging in our assumption on $E_{m}$, we have

$$
E_{m+1}=M E_{m}+M E_{0} C^{m}=M E_{0} C^{m}+M\left(\sum_{j=1}^{m} M^{j} E_{0} C^{m-j}\right)=\sum_{j=1}^{m+1} M^{j} E_{0} C^{m+1-j}
$$

To bound the extrapolation error,

$$
\sum_{m=1}^{s} E_{m}=\sum_{m=1}^{s}\left(\sum_{j=1}^{m} M^{j} E_{0} C^{m-j}\right)=\sum_{\ell=0}^{s-1}\left(\sum_{j=1}^{s-\ell} M^{j}\right) E_{0} C^{\ell}=\sum_{\ell=1}^{s} M^{\ell} E_{0}\left(\sum_{i=0}^{s-\ell} C^{i}\right)
$$

Therefore,

$$
\left\|\bar{z}_{k, s}-z^{\star}\right\| \leq\left\|z_{k+s}-z^{\star}\right\|+\sum_{\ell=1}^{s}\left\|M^{\ell}\right\|\left\|E_{0}\right\|\left\|\sum_{i=0}^{s-\ell} C_{(1,1)}^{i}\right\|
$$

In the case of $s=+\infty$, we have

$$
\left\|\bar{z}_{k, \infty}-z^{\star}\right\| \leq \sum_{\ell=1}^{\infty}\left\|M^{\ell}\right\|\left\|E_{0}(\operatorname{Id}-C)_{(:, 1)}^{-1}\right\|=\frac{\left\|E_{0}\right\|}{1-\sum_{i} c_{i}} \sum_{\ell=1}^{\infty}\left\|M^{\ell}\right\|
$$

The fact that $B_{s}$ is uniformly bounded in $s$ if $\rho(M)<1$ and $\rho(C)<1$ follows because this implies that $\sum_{\ell=1}^{\infty}\left\|M^{\ell}\right\|<\infty$ thanks to the Gelfand formula, and $\sum_{i=0}^{\infty} C^{i}=(\operatorname{Id}-C)^{-1}$ and its $(1,1)^{t h}$ entry is precisely $\frac{1}{1-\sum_{i} c_{i}}$.

To control the coefficients fitting error $\epsilon_{k}$, we follow closely the arguments of Section 6.7 in [38], since this amounts to understanding the behaviour of the coefficients $c_{k}$, which are precisely the MPE coefficients. Recall our assumption that $M$ is diagonalisable, so $M=U^{\top} \Sigma U$ where $U$ is an orthogonal matrix and $\Sigma$ is a diagonal matrix with the eigenvalues of $M$ as its diagonal. Then, letting $u_{k} \stackrel{\text { def }}{=} U v_{k}$,

$$
\begin{aligned}
\epsilon_{k} & =\min _{c \in \mathbb{R}^{q}}\left\|\sum_{i=1}^{q} c_{i} v_{k-i}-v_{k}\right\| \\
& =\min _{c \in \mathbb{R}^{q}}\left\|\sum_{i=1}^{q} c_{i} \Sigma^{k-i} u_{0}-\Sigma^{k} u_{0}\right\|=\min _{g \in \mathcal{P}_{q}}\left\|\Sigma^{k-q} g(\Sigma) u_{0}\right\| \leq\left\|u_{0}\right\| \min _{g \in \mathcal{P}_{q}} \max _{z \in \lambda(M)}|z|^{k-q}|g(z)|
\end{aligned}
$$

where $\mathcal{P}_{q}$ is the set of monic polynomials of degree $q$ and $\lambda(M)$ is the spectrum of $M$. Choosing $g=\prod_{j=1}^{q}\left(z-\lambda_{j}\right)$, we have $g\left(\lambda_{j}\right)=0$ for $j=1, \ldots, q$, so

$$
\begin{equation*}
\epsilon_{k} \leq\left\|u_{0}\right\|\left|\lambda_{q+1}\right|^{k-q} \max _{\ell>q} \prod_{j=1}^{q}\left|\lambda_{j}-\lambda_{\ell}\right| \tag{C.5}
\end{equation*}
$$

The claim that $\rho\left(C_{k}\right)<1$ holds since the eigenvalues of $C$ are precisely the roots of the polynomial $Q(z)=z^{k-1}-$ $\sum_{i=1}^{k-1} c_{j} z^{k-1-i}$, and from [38], if $\left|\lambda_{q}\right|>\left|\lambda_{q+1}\right|$, then $Q$ has precisely $q$ roots $r_{1}, \ldots, r_{q}$ satisfying $r_{j}=\lambda_{j}+\mathcal{O}\left(\left|\lambda_{q+1} / \lambda_{j}\right|^{k}\right)$. So, $\left|r_{j}\right|<1$ for all $k$ sufficiently large. To prove the non-asymptotic bounds on $\epsilon_{k}$, first observe that $z_{k+1}-z_{k}=$ $M\left(z_{k}-z_{k-1}\right)$ implies $z_{k+1}-z^{\star}=M\left(z_{k}-z_{*}\right)$ and $z_{k+1}-z_{k}=(M-\mathrm{Id})\left(z_{k}-z^{\star}\right)$. So, letting $\gamma_{i}=-c_{k, i} /\left(1-\sum_{i} c_{k, i}\right)$ for $i=1, \ldots, q$ and $\gamma_{0}=1 /\left(1-\sum_{i} c_{k, i}\right)$, we have

$$
\begin{equation*}
\frac{1}{1-\sum_{i} c_{k, i}}\left(v_{k}-\sum_{i=1}^{q} c_{k, i} v_{k-i}\right)=\sum_{i=0}^{q} \gamma_{i} v_{k-i}=(M-\mathrm{Id}) \sum_{i=0}^{q} \gamma_{i}\left(z_{k-i-1}-z^{\star}\right) \tag{C.6}
\end{equation*}
$$

Now, $y \stackrel{\text { def }}{=} \sum_{i=0}^{q} \gamma_{i} z_{k-i-1}$ is precisely the MPE update and norm bounds on this are presented in [38]. For completeness, we reproduce their arguments here: Let $A \stackrel{\text { def }}{=} \mathrm{Id}-M$, by our assumption of $\lambda(M) \subset(-1,1)$, we have that $A$ is positive definite. Then,

$$
\begin{aligned}
\left\|A^{1 / 2}\left(y-z^{\star}\right)\right\|^{2} & =\left\langle A\left(y-z^{\star}\right),\left(y-z^{\star}\right)\right\rangle \\
& =-\left\langle\sum_{i=0}^{q} \gamma_{i} v_{k-i},\left(y-z^{\star}\right)+w\right\rangle
\end{aligned}
$$

where $w=\sum_{j=1}^{q} a_{j} v_{k-j}$ with $a \in \mathbb{R}^{q}$ being arbitrary, since by definition of $\gamma,\left\langle\sum_{i=0}^{q} \gamma_{i} v_{k-i}, v_{\ell}\right\rangle=0$ for all $\ell=$ $k-q, \ldots, k-1$. We can write

$$
w=\sum_{j=1}^{q} a_{j}(M-\mathrm{Id})\left(z_{k-j-1}-z^{\star}\right)=\sum_{j=1}^{q} a_{j}(M-\mathrm{Id}) M^{k-j-1}\left(z_{0}-z^{\star}\right)=f(M)\left(z_{0}-z^{\star}\right)
$$

where $f(z)=z^{k-q-1}(z-1) \sum_{j=1}^{q} a_{j} z^{q-j}$, and we can write

$$
y-z^{\star}=\sum_{i=0}^{q} \gamma_{i} M^{k-i-1}\left(z_{0}-z^{\star}\right)=g(M)\left(z_{0}-z^{\star}\right)
$$

where $g(z)=z^{k-q-1} \sum_{i=0}^{q} \gamma_{i} z^{q-i}$. Therefore, $f(z)+g(z)=z^{k-1-q} h(z)$, where $h$ is a polynomial of degree $q$ such that $h(1)=1$. Moreover, since the coefficients $a_{j}$ are arbitrary, $h$ can be considered as an arbitrary element of $\tilde{\mathcal{P}}_{q}$, the set of all polynomials of degree $q$ such that $h(1)=1$. Therefore

$$
\begin{aligned}
\left\|A^{-1 / 2}\left(y-z^{\star}\right)\right\|^{2} & \leq\left\|A^{-1 / 2}\left(y-z^{\star}\right)\right\| \min _{h \in \tilde{\mathcal{P}}_{q}}\left\|M^{n} h(M)\left(z_{0}-z^{\star}\right)\right\| \\
& \leq\left\|A^{-1 / 2}\left(y-z^{\star}\right)\right\| \min _{h \in \tilde{\mathcal{P}}_{q}} \max _{t \in \lambda(M)}\left|t^{n} h(t)\right|\left\|z_{0}-z^{\star}\right\|
\end{aligned}
$$

In particular, combining this with (C.6), we have

$$
\frac{\epsilon_{k}}{\left|1-\sum_{i} c_{k, i}\right|} \leq\left\|z_{0}-z^{\star}\right\|\left\|(\operatorname{Id}-M)^{1 / 2}\right\| \rho(M)^{n} \min _{h \in \tilde{\mathcal{P}}_{q}} \max _{t \in \lambda(M)}|h(t)|
$$

Finally, in our case where $\lambda(M)=[\alpha, \beta]$ with $1>\beta>\alpha>-1$, it is well known that $\min _{h \in \tilde{\mathcal{P}}_{q}} \max _{t \in \lambda(M)}|h(t)|$ has an explicit expression (see, for example, [9] or [38, Section 7.3.1]):

$$
\min _{h \in \tilde{\mathcal{P}}_{q}} \max _{z \in \lambda(M)}|h(z)| \leq \max _{z \in \lambda(M)}\left|h_{*}(z)\right|
$$

where $h_{*}(z) \stackrel{\text { def }}{=} \frac{T_{q}\left(\frac{2 z-\alpha-\beta}{\beta-\alpha}\right)}{T_{q}\left(\frac{2-\alpha-\beta}{\beta-\alpha}\right)}$ where $T_{q}(x)$ is the $q^{\text {th }}$ Chebyshev polynomial and it is well known that

$$
\begin{equation*}
\min _{h \in \tilde{\mathcal{P}}_{q}} \max _{z \in[\alpha, \beta]}|h(z)| \leq 2\left(\frac{\sqrt{\eta}-1}{\sqrt{\eta}+1}\right)^{q} \tag{C.7}
\end{equation*}
$$

where $\eta=\frac{1-\alpha}{1-\beta}$.


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[^1]:    ${ }^{1}$ https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/

