# Compressed Sensing with Fourier Measurements

November 8, 2017

# Random Gaussian matrices

Given $x \in \mathbb{C}^{N \times N}$, let $\hat{x} = U_{\mathrm{Haar}} \hat{z}$ where

$$\hat{z} \in \operatorname*{argmin}_{z \in \mathbb{C}^{N \times N}} \|z\|_1 \text{ subject to } \left\| A U_{\mathrm{Haar}}^{-1} z - y \right\|_2 \leq \eta.$$

$x$

$\hat{x}$

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Problem: No fast transforms for Gaussian/Bernoulli random matrices, and they are expensive to store. E.g. To store a Gaussian matrix for sampling $512 \times 512$ images at 5% subsampling would require 25GB.

# Fourier Sampling in Inverse Problems

Many imaging problems* are modelled by the Fourier transform

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \omega \cdot x} \, dx,$$

or the Radon transform $\mathcal{R}f : \boldsymbol{S}^{d-1} \times \mathbb{R} \to \mathbb{C}$ (where $\boldsymbol{S}^{d-1}$ denotes the sphere)

$$\mathcal{R}f(\theta, p) = \int_{\langle x, \theta \rangle = p} f(x) \, dm(x),$$

where $dm$ denotes Lebesgue measure on the hyperplane $\{x : \langle x, \theta \rangle = p\}$.

- Fourier slice theorem $\Rightarrow$ both problems can be viewed as the problem of reconstructing $f$ from pointwise samples of its Fourier transform.

$$g = \mathcal{F}f, \quad f \in L^2(\mathbb{R}^d). \tag{1}$$

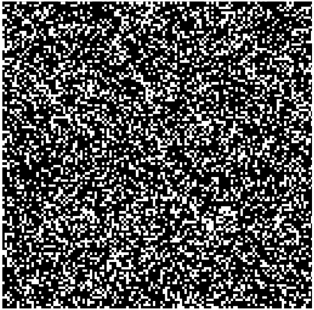
* Magnetic Resonance Imaging (MRI), X-ray Computed Tomography, Electron Microscopy, Radio interferometry, ...

# Random Fourier sampling and the RIP

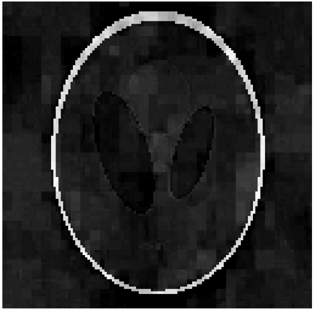
M. Rudelson and R. Vershynin (2008) proved that $P_\Omega U_{df}$ satisfies $\delta_s(P_\Omega U_{df}) \leq \delta$ with high probability provided that $\Omega$ consists of $m$ indices chosen uniformly at random such that

$$m \geq C\delta^{-2} s \ln^4(N).$$

$\Omega$ (30%)



$\hat{x}$ (128 $\times$ 128)



NB: $\ln^4(N)$ is large in practice. In our case, we are interested in $128 \times 128$ images, if $s = 0.05N^2$ then $s \ln^4(N^2) \approx 7.26 \times 10^6 > N^2$.

# Uniform vs Nonuniform recovery

The results we have seen so far are uniform recovery guarantees: *Under certain conditions, with high probability, we can recover all $s$-sparse vectors.*

There is another kind of statement, namely nonuniform recovery guarantees: *Let $x$ be a fixed sparse vector. Then, under certain conditions, with high probability, $x$ can be recovered.*

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Theorem (Candés & Plan, 2011): *Let $U \in \mathbb{C}^{N\times N}$ be an isometry. If $\alpha$ is s-sparse, then solving*

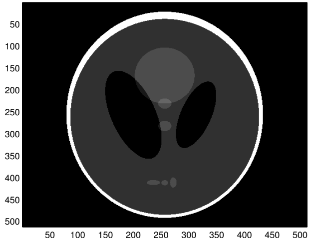
$$\min_{\beta\in\mathbb{C}^N} \|\beta\|_1 \text{ subject to } P_\Omega U\beta = P_\Omega U\alpha$$

*recovers $\alpha$ with probability exceeding $(1-\epsilon)$ if $\Omega$ is chosen uniformly at random such that it is of cardinality*

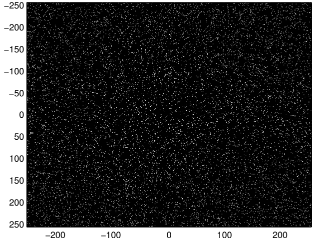
$$m \gtrsim s \cdot \log(N/\epsilon) \cdot \mu(U) \cdot N$$

*where the coherence is* $\mu(U) := \max_{j,k=1,\ldots,N} |U_{j,k}|^2 \in [N^{-1}, 1]$.

# An intriguing experiment (Candès, Romberg & Tao, 2006)




x




Ω (4.24%)

$$\min_{z\in\mathbb{C}^{N\times N}} \|Dz\|_1 \text{ subject to } P_\Omega Uz = P_\Omega Ux$$

- $Uz = \left(N^{-1}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N} z_{j_1,j_2} e^{i2\pi(k_1 j_1 + k_2 j_2)}\right)_{k_1,k_2=-\lfloor N/2\rfloor,\ldots,\lceil N/2\rceil-1}.$
- $Dz = D_1 z + iD_2 z$ where

$$D_1 z = (x_{k+1,j} - z_{k,j})_{k,j=1}^{N}, \quad D_2 z = (z_{k,j+1} - z_{k,j})_{k,j=1}^{N}$$

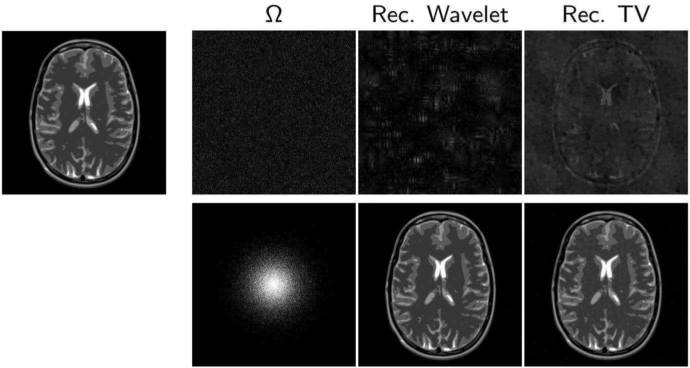
with $z_{N+1,j} := z_{1,j}$ and $z_{k,N+1} := z_{k,1}$.

# Variable Density Sampling

## 5% sampling for $1024 \times 1024$ phantom

In the Fourier-wavelets case: $\mu(U_{dft} U_{dwt}^{-1}) = 1$.

Lustig, Donoho & Pauli '07, Lustig et al. '08: Sample more densely at low Fourier frequencies and less at higher Fourier frequencies.




Why does VDS work?

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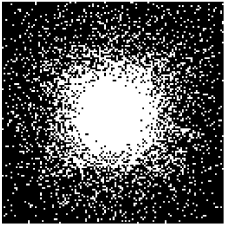
Test phantom constructed by Guerquin-Kern, Lejeune, Pruessmann, Unser, 2012

# Sparsity and the Flip Test

*In standard CS, the only signal structure considered is sparsity. In contrast, the flip test will demonstrate that we must look beyond sparsity.*

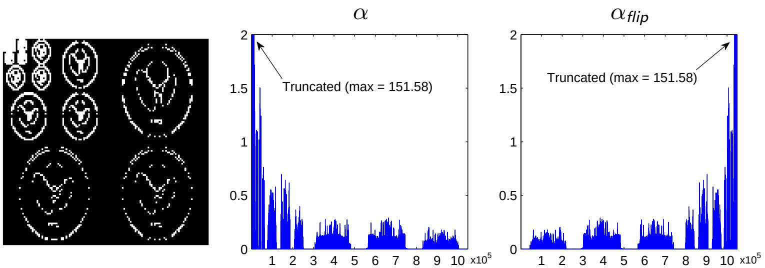
Consider the reconstruction of $x$ from $P_\Omega U_{df} x$ by solving

$$\min_z \|z\|_1 \text{ subject to } P_\Omega U_{df} U_{dw}^{-1} z = P_\Omega U_{df} x.$$



# Sparsity and the Flip Test

Let $\alpha$ be the wavelet coefficients of $x$. Let $\alpha^{flip} = (\alpha_N, \ldots, \alpha_1)$ and $x^{flip} = U_{dw}^{-1}\alpha^{flip}$.

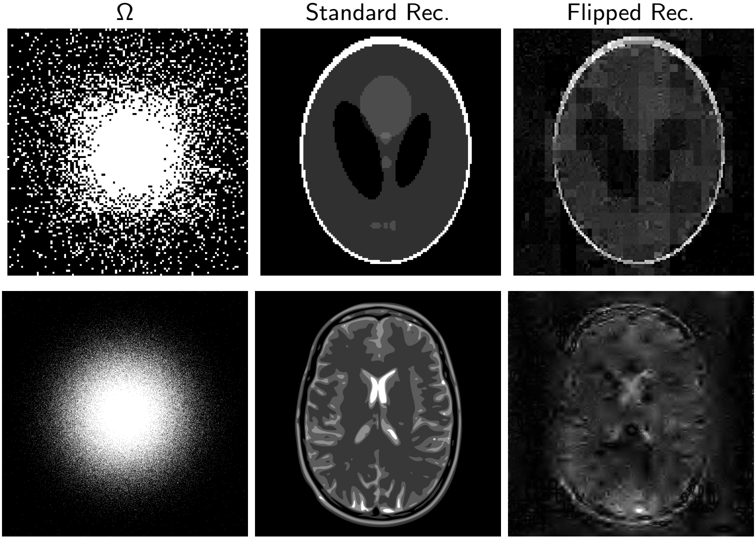



For the same $\Omega$, let

$$\tilde{\alpha} \in \arg\min_{\beta} \|\beta\|_1 \text{ subject to } P_\Omega U_{df} U_{dw}^{-1}\beta = P_\Omega U_{df} x^{flip}$$

If sparsity was enough, then the wavelet coefficients $\tilde{\alpha}$ should be $\alpha^{flip}$, we should be able to recover $x$ as $U_{dw}^{-1}\tilde{\alpha}^{flip}$.

# Sparsity and the Flip Test



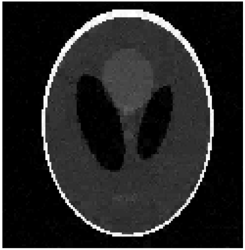
# Flip test for random Gaussian measurements

In the Fourier-wavelets case, we saw that $\Omega$ cannot depend on sparsity alone.

Actually, variable density sampling patterns exploit the fact that natural images are asymptotically sparse in wavelets.

However, random Gaussian measurements are insensitive to sparsity structure:

Standard Rec.



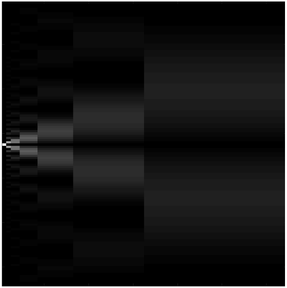
Flipped Rec.

# Asymptotic Incoherence

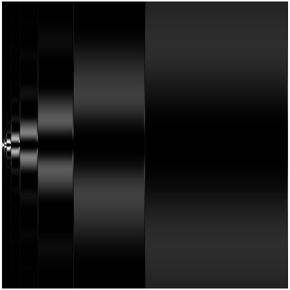
If $U = U_{df} U_{dw}^{-1}$ is the Fourier-wavelets matrix, then

$$\mu(P_N^{\perp} U), \mu(U P_N^{\perp}) = \mathcal{O}\left(N^{-1}\right).$$

Fourier to Haar



Fourier to DB4



Implication: Sample more at low Fourier frequencies where the local coherence is high and less at higher Fourier frequencies.

# Recovery of Wavelet Coefficients from Partial Fourier Data

Theorem:

- $\{N_k\}_{k=1}^r$ and $\{M_k\}_{k=1}^r$ correspond to wavelet scales.
- The mother wavelet $\psi$ has $v$ vanishing moments.
- There exists $\alpha \geq 1$, $C > 0$ such that $\left|\hat{\psi}(\xi)\right| \leq \frac{C}{(1+|\xi|)^\alpha}$ for all $\xi \in \mathbb{R}$.

One is guaranteed recovery of a wavelet sequence which is $s_k$-sparse in the $k^{th}$ wavelet scale by choosing $\Omega = \Omega_1 \cup \cdots \cup \Omega_r$ where $\Omega_k \subset \{N_{k-1}+1, \ldots, N_k\}$ are $m_k$ samples chosen uniformly at random, with

$$\frac{m_k}{N_k - N_{k-1}} \gtrsim \frac{1}{N_{k-1}} \cdot \mathcal{L} \cdot \left( \hat{s}_k + \sum_{l=1}^{k-2} s_j \cdot 2^{-\alpha(k-l)} + \sum_{l=k+2}^{r} s_l \cdot 2^{-v(l-k)} \right)$$

where $\hat{s}_k = \max\{s_{k-1}, s_k, s_{k+1}\}$ and $\mathcal{L} = \log(\epsilon^{-1}) \cdot \log\left(KN\sqrt{s}\right)$

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*Breaking the coherence barrier: A new theory for compressed sensing.* Adcock, Hansen, Poon & Roman (2013)

# Resolution Dependence (5% samples, varying resolution)

Asymptotic sparsity and asymptotic incoherence are only witnessed when $N$ is large. Thus, V. D. sampling only reaps their benefits for large values of $N$ and the success of compressed sensing is resolution dependent.

256x256

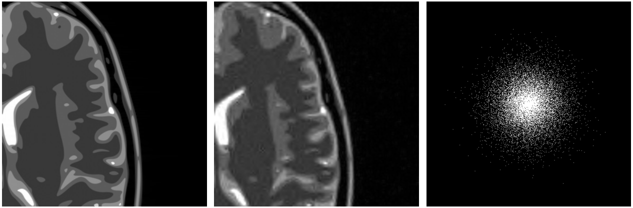
Error:

19.86%

512x512

Error:

10.69%



# Resolution Dependence (5% samples, varying resolution)

1024x1024

Error:

7.35%

2048x2048

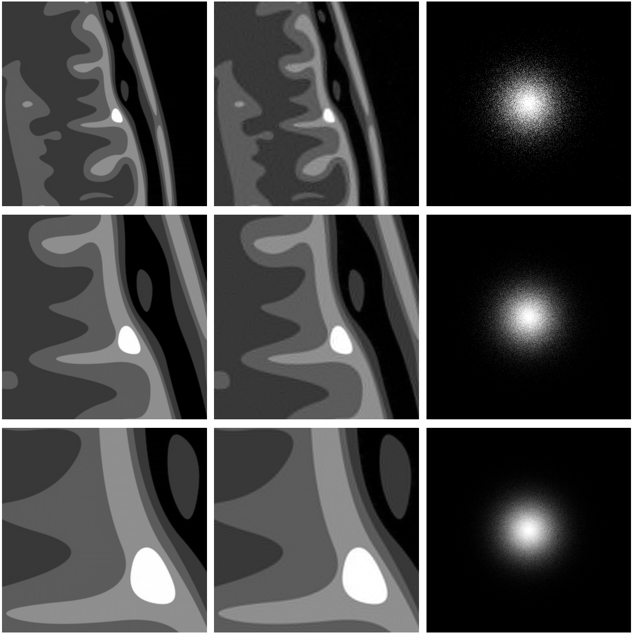
Error:

4.87%

4096x4096

Error:

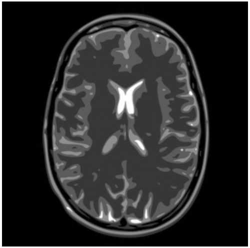
3.06%



# Recovering Fine Details

At finer wavelet scales, the presence of sparsity and incoherence with Fourier samples allows us to subsample. Thus, compressed sensing allows one to enhance fine details without increasing the number of samples.

In the next example, consider the reconstruction of a $2048 \times 2048$ test phantom with details added at the finest wavelet scale.



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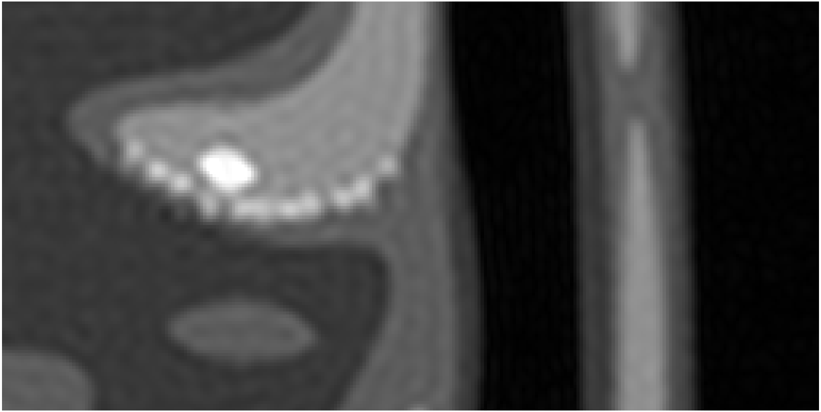

Figure: 2048 × 2048 linear reconstruction from the first 512 × 512 Fourier samples (6.25%)

# Recovering Fine Details


Figure: 2048 $\times$ 2048 reconstruction from a multilevel scheme using 512 $\times$ 512 Fourier samples (6.25%)